Appendix A

Shock-Wave Geometries

A.1 More on AdS shock-wave solutions

In this appendix we give some details of the geodesics and stress-energy tensor of massless particles in AdS, discussed in chapters 2-3 and their properties.

We write AdS space in the co-ordinate system (2.12), $y^\mu = (u, v, y^i)$, i running from 1 to $d-2$. The metric reads:

$$\mathrm{d}s^2 = \frac{4}{\Omega^2}\,\eta_{\mu\nu}\mathrm{d}y^\mu \mathrm{d}y^\nu, \tag{A.1}$$

where the conformal factor is given by $\Omega = 1 - y^2/\ell^2$.

It is well-known that the null geodesics of two conformally related space-times are the same, up to a reparametrisation of the geodesic length. Therefore, null trajectories in the above co-ordinates will take the same form as those in Minkowski space. It is nevertheless convenient for the computation of the stress-energy tensor to see explicitly how the affine parameter changes.

The geodesic equation and the mass-shell condition give:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{\eta_{\mu\nu}\dot{z}^\nu}{\Omega^2}\right) &= 2\frac{\eta_{\mu\nu}z^\nu \mathcal{L}}{\ell^2\,\Omega} \\ \mathcal{L} &= \frac{1}{\Omega^2}\,\eta_{\mu\nu}\dot{z}^\mu\dot{z}^\nu = 0. \end{aligned} \tag{A.2}$$

$\mathcal{L}$ is the Lagrange density, defined by the second of (A.2), and λ the affine parameter along the geodesic. These equations integrate to

$$\eta_{\mu\nu}\,\dot{z}^\nu = v_\mu\,\Omega^2. \tag{A.3}$$

v_μ is a constant, lightlike vector satisfying $\eta^{\mu\nu}v_\mu v_\nu = 0$ to be determined by the boundary conditions. This equation also relates the affine parameter in AdS to the affine parameter in Minkowski space.

The stress-energy tensor (2.5) now equals:

$$T_{\mu\nu} = -p\,\Omega^d\,v_\mu v_\nu \int \mathrm{d}s\,\delta^{(d)}(y - z(s)), \tag{A.4}$$

and choosing co-ordinates where momentum is purely in the v-direction, this reduces to:

$$T_{uu} \quad = \quad -p\,\Omega^d\,\delta(u-u_0)\,\delta(\rho-\rho_0), \tag{A.5}$$

where $\rho = \sum_{i=1}^{d-2} y_i^2$. Notice that in order for the metric (3.56) to be a solution of Einstein's equations with this stress-energy tensor, we need the initial condition $u_0 = 0$. It is also convenient to take $\rho_0 = 0$. Thus we get the stress-energy tensor used in chapter 3,

$$T_{uu} = -p\,\delta(u)\delta(\rho), \tag{A.6}$$

which gives rise to the delta-function in (3.57). This form for the stress-energy tensor agrees with the one computed in chapter 4, equation (4.98), in Poincare co-ordinates, and for the case $g_{(0)ij} = \eta_{ij}$. One can check this by performing the following co-ordinate transformation from y^μ to Poincare co-ordinates $x^\mu = (r, t, x^i)$:

$$\begin{aligned}
u &= \frac{t^2 - r^2 - \vec{x}^2}{r+t} \\
v &= \frac{\ell^2}{r+t} \\
y^i &= \frac{\ell x^i}{r+t} \\
\Omega &= \frac{2r}{r+t} \\
r &= \frac{1}{2v}\,(\ell^2 - uv - \rho^2) \\
t &= \frac{1}{2v}\,(\ell^2 + uv + \rho^2) \\
x^i &= \frac{\ell}{v}\,y^i.
\end{aligned} \tag{A.7}$$

With (A.6) at hand, one can compute the back-reaction on the AdS metric, obtaining the solution found by Horowitz and Itzhaki with the shift functions as given in (3.60). The next step is then to compute the geodesics of a test particle in the back-reaction corrected metric. The computation goes along the same lines as the one above. We do not give the details here since it is a straightforward exercise, but give only the results. We concentrate on trajectories whose initial velocities are perpendicular to the velocity of the shockwave, that is, the geodesics with $v = y^i = 0$ before the collision. This gives a head-on collision.

It turns out that the geodesic equations can again be exactly integrated, and the effect is the same as in Minkowski space: there is a shift in the v co-ordinate and a deflection in the x^i-plane which nevertheless is negligible in the eikonal approximation where the impact parameter is much larger than the Planck length. In this approximation, the shift is given by

$$\delta v = -8\pi G_{\rm N}\,p_u\,F_0\,\theta(u), \tag{A.8}$$

where F_0 is the shift function before the collision, $F_0 = F(u=0)$.

Of course the same results can be found from geodesics in Minkowski space by noting that massless geodesics are invariant under conformal transformations of the metric.

It is interesting to note that, when one considers only one particle, there is no self-interaction, and therefore the present solution to the Einstein-matter system with the given boundary conditions is exact. However, when considering two particles this is no longer true, and one has to restrict oneself to consider a "soft" test particle in the background of a "hard" particle.

A.2 The induced two-dimensional Ricci tensor

In this Appendix we outline the proof that Einstein's equations with a massless source reduce to the conditions (2.9)-(2.10). We also compute the curvature of the transverse part of the metric, equation (2.32). This computation follows [40], and for more details we refer to that paper.

The ansatz in [40] for the metric is the following:

$$\mathrm{d}\hat{s}^2 = 2A(\hat{u},\hat{v})\,\mathrm{d}\hat{v}(\mathrm{d}\hat{u} - \delta(v)\mathrm{d}\hat{v}) + g(\hat{u},\hat{v})\,h_{ij}(\hat{x}^i)\mathrm{d}\hat{x}^i\mathrm{d}\hat{x}^j. \tag{A.9}$$

We also have the unperturbed metric

$$\mathrm{d}s^2 = 2A(u,v)\,\mathrm{d}u\mathrm{d}v + g(u,v)\,h_{ij}(x^i)\mathrm{d}x^i\mathrm{d}x^j, \tag{A.10}$$

which will be assumed to solve Einstein's equations. (A.9) is related to (A.10) by a shift *and* a co-ordinate transformation:

$$\begin{aligned} \hat{u} &= u + \theta f \\ \hat{v} &= v \\ \hat{x}^i &= x^i. \end{aligned} \tag{A.11}$$

The metric (A.9) should be a solution of Einstein's equations with a massless source:

$$\begin{aligned} R_{\mu\nu}[\hat{G}] &= R_{\mu\nu}[G] + \delta R_{\mu\nu}[G] = -8\pi G_{\mathrm{N}}\hat{T}_{\mu\nu} \\ R_{\mu\nu}[G] &= 0 \\ T^{\hat{u}\hat{u}} &= 4p\,\delta(\hat{v})\,\delta(\tilde{x}), \end{aligned} \tag{A.12}$$

so our massless particle travels along the null geodesic $\hat{v} = 0$, $\hat{x}^i = 0$.

Let us first work out the vacuum piece of Einstein's equations, $R_{\mu\nu}[G] = 0$. We use the formula

$$R^{\mu}{}_{i\mu j}[G] = \frac{1}{\sqrt{-G}}\,\partial_\mu\left(\sqrt{-G}\,\Gamma^\mu_{ij}\right) - \partial_i\partial_j\left(\log\sqrt{-G}\right) - \Gamma^\mu_{\nu i}\Gamma^\nu_{j\mu}, \tag{A.13}$$

and we have:

$$\begin{aligned} \sqrt{-G} &= Ag\sqrt{h} \\ \Gamma^\alpha_{ij} &= -\frac{1}{2}\,g^{\alpha\beta}h_{ij}\partial_\beta g \end{aligned}$$

$$\begin{aligned}
\Gamma^i_{j\alpha} &= \frac{1}{2g}\,\delta^i_j\partial_\alpha g \\
\Gamma^i_{\alpha j} &= \frac{1}{2g}\,\delta^i_j\partial_\alpha g \\
\Gamma^\alpha_{\beta i} &= \Gamma_{i\alpha\beta} = 0,
\end{aligned} \tag{A.14}$$

where the indices μ, ν run from 1 to 4, α and β take the values 1, 2, and i, j take the values $3, 4$. Plugging this in equation (A.13), we get:

$$\begin{aligned}
R_{ij}[G] &= R_{ij}[h] - \frac{1}{2Ag}\,h_{ij}\partial_\alpha\left(Agg^{\alpha\beta}\partial_\beta g\right) - \Gamma^\alpha_{ki}\Gamma^k_{j\alpha} - \Gamma^k_{\alpha i}\Gamma^\alpha_{jk} \\
&= R_{ij}[h] - \frac{1}{A}\,h_{ij}\partial_u\partial_v g = 0.
\end{aligned} \tag{A.15}$$

Here $R_{ij}[h]$ is the two-dimensional Ricci tensor calculated in the metric h_{ij}. This gives

$$R_{ij}[h] = \frac{1}{A}\,\partial_u\partial_v g\,h_{ij}, \tag{A.16}$$

which gives (2.32).

After some algebra, and using the vacuum solutions, one finds that the remaining piece of the metric only contributes the $\hat{u}\hat{u}$-component of the Ricci tensor. Einstein's equations,

$$\delta R_{\mu\nu}[G] = -8\pi G_{\mathrm{N}}\hat{T}_{\mu\nu}, \tag{A.17}$$

are then satisfied provided $R_{\mu\nu}[G] = 0$ and (2.9)-(2.10) hold. More details can be found in the appendices of [40].