Appendix B

Scaling and Classical Solutions of the Einstein-Hilbert Action

B.1 Scaling of curvature

In this section we give the details of the final rescaled (up to lowest order in $h_{i\alpha}$, all orders in ϵ) Ricci tensor $R_{\mu\nu}$. From here one can simply check the expansion of the Einstein-Hilbert action[1]. Higher orders in $h_{i\alpha}$ (quadratic at $1/\epsilon^2$ and at ϵ^0) are not necessary as we are not going to consider the fluctuations of the metric.

Under the rescaling (3.3), the Christoffel symbols transform as:

$$\begin{aligned}
\Gamma^{\gamma}_{\alpha\beta}(G) &= \Gamma^{\gamma}_{\alpha\beta}(\hat{G}) + \frac{1-\epsilon}{2}\, g^{i\gamma}\partial_i g_{\alpha\beta} \\
\Gamma^{\alpha}_{ij}(G) &= \frac{1}{\epsilon}\,\Gamma^{\alpha}_{ij}(\hat{G}) + \frac{\epsilon-1}{2\epsilon^2}\, g^{\alpha\beta}\partial_\beta g_{ij} \\
\Gamma^{\alpha}_{\beta i}(G) &= \Gamma^{\alpha}_{\beta i}(\hat{G}) + \frac{1-\epsilon}{2\epsilon}\,(g^{\alpha\gamma}\partial_\beta g_{\gamma i} - g^{\alpha\gamma}\partial_\gamma g_{\beta i} + g^{\alpha j}\partial_\beta g_{ij}) \\
\Gamma^{k}_{ij}(G) &= \Gamma^{k}_{ij}(\hat{G}) + \frac{\epsilon-1}{2\epsilon}\, g^{k\alpha}\partial_\alpha g_{ij} \\
\Gamma^{i}_{\alpha\beta}(G) &= \epsilon\,\Gamma^{i}_{\alpha\beta}(\hat{G}) + \frac{\epsilon(1-\epsilon)}{2}\, g^{ij}\partial_j g_{\alpha\beta} \\
\Gamma^{i}_{\alpha j}(G) &= \Gamma^{i}_{\alpha j}(\hat{G}) + \frac{\epsilon-1}{2}\,(g^{i\beta}\partial_j g_{\alpha\beta} + g^{ik}\partial_j g_{\alpha k} - g^{ik}\partial_k g_{\alpha j})
\end{aligned} \qquad \text{(B.1)}$$

where $G_{\mu\nu}$ is the ϵ-dependent metric, whereas $\hat{G}_{\mu\nu}$ is the rescaled metric, which is independent of ϵ.

Working out the curvature components, we get:

$$R_{\alpha\beta}[G] = \epsilon^0(R_{\alpha\beta}[\hat{G}] - \frac{1}{2}\nabla_\beta(g^{ik}\partial_\alpha g_{ik}) - \frac{1}{4}g^{ij}\partial_\alpha g_{kj}g^{km}\partial_\beta g_{im}) +$$

[1]For notational simplicity, we denote the transverse metric by g_{ij}. In chapter 3 it is denoted by h_{ij}.

$$+\quad \epsilon^2(-\frac{1}{2}\nabla_i(g^{ij}\partial_j g_{\alpha\beta}) - \frac{1}{4}g^{\gamma\rho}\partial_i g_{\gamma\rho}g^{ij}\partial_j g_{\alpha\beta} +$$
$$+\quad \frac{1}{4}g^{\gamma\rho}\partial_k g_{\beta\rho}g^{ki}\partial_i g_{\alpha\gamma} + \frac{1}{4}g^{\gamma\rho}\partial_k g_{\alpha\rho}g^{ki}\partial_i g_{\beta\gamma}) \quad \text{(B.2)}$$

The leading term in $R_{i\alpha}$ is at zero order in $h_{i\alpha}$ which is already sufficient for our purposes as it is always multiplied by $h_{i\alpha}$ in the action and this term arises at order ϵ^0

$$\begin{aligned} R_{i\alpha} &= \epsilon^0(\frac{1}{2}\nabla_\beta(g^{\beta\rho}\partial_i g_{\alpha\rho}) - \frac{1}{2}\nabla_\alpha(g^{\beta\rho}\partial_i g_{\beta\rho}) + \frac{1}{2}\nabla_k(g^{kj}\partial_a g_{ij}) \\ &- \frac{1}{2}\nabla_i(g^{kj}\partial_\alpha g_{kj}) + \frac{1}{4}g^{\gamma\rho}\partial_i g_{\alpha\rho}g^{kj}\partial_\gamma g_{kj} + \frac{1}{4}g^{km}\partial_\alpha g_{im}g^{\gamma\beta}\partial_k g_{\gamma\beta} \\ &- \frac{1}{2}g^{\beta\rho}\partial_\rho g_{ik}g^{kj}\partial_j g_{\alpha\beta}) \end{aligned} \quad \text{(B.3)}$$

R_{ij} is identical to $R_{\alpha\beta}$ under the interchange of Greek and Roman indices and $\epsilon \to \epsilon^{-1}$.

$$\begin{aligned} R_{ij}[G] &= \epsilon^{-2}(-\frac{1}{2}\nabla_\alpha(g^{\alpha\beta}\partial_\beta g_{ij}) - \frac{1}{4}g^{km}\partial_\alpha g_{km}g^{\alpha\beta}\partial_\beta g_{ij} \\ &+ \frac{1}{4}g^{km}\partial_\gamma g_{jm}g^{\gamma\alpha}\partial_\alpha g_{ik} + \frac{1}{4}g^{km}\partial_\gamma g_{im}g^{\gamma\alpha}\partial_\alpha g_{jk}) \\ &+ \epsilon^0(R_{ij}[\hat{G}] - \frac{1}{2}\nabla_j(g^{\alpha\gamma}\partial_i g_{\alpha\gamma}) - \frac{1}{4}g^{\alpha\beta}\partial_\alpha g_{\gamma\beta}g^{\gamma\rho}\partial_j g_{\alpha\rho}) \end{aligned} \quad \text{(B.4)}$$

B.2 Scaling of the exterior curvature

The exterior curvature part of the Einstein – Hilbert action is

$$S = \frac{1}{\ell_{\rm Pl}^{d-2}}\int \sqrt{\gamma}\,\nabla_\mu n^\mu \quad \text{(B.5)}$$

γ is the boundary metric which under rescaling is multiplied by $\ell_\parallel^2\ell_\perp^{2(d-2)}$. The normal n will have a non-zero component only in the direction perpendicular to the boundary, parallel to the longitudinal scattering plane. Thus as the longitudinal metric scales with $\ell_\parallel^2$ the normalisation condition for n implies that it will also scale with $\ell_\parallel$. Thus,

$$\nabla_\mu n^\mu = \frac{\nabla_\alpha n^\alpha}{\ell_\parallel} + \frac{\nabla_i n^i}{\ell_\perp}. \quad \text{(B.6)}$$

The exterior curvature term of the action becomes

$$\epsilon^{d-4}S_{\partial M} = \frac{1}{\epsilon^2}\int \sqrt{\gamma}\,\nabla_\alpha n^\alpha + \frac{1}{\epsilon}\int \sqrt{\gamma}\,\nabla_i n^i. \quad \text{(B.7)}$$

As claimed in the text there is no additional contribution to the boundary action coming from the exterior curvature.

B.3 Classical solutions

In this appendix we give some more details on how to solve the equations of motion for the background, coming from the $\frac{1}{\epsilon^2}$ part of the action (3.10).

We rewrite the action (3.12) in the following form (now concentrating on the two-dimensional covariant part),

$$S = -\frac{1}{2}\int \sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \Lambda\phi^2 - \frac{1}{2}\phi^2 R[g]\right). \tag{B.8}$$

This action belongs to the class of actions considered in [14], with Lagrangian of the form

$$L = \sqrt{-g}\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - \lambda\phi^{2k} - Q\phi^2 R[g]\right), \tag{B.9}$$

with the obvious values $k = 1$, $\lambda = -\frac{(d-2)}{2(d-3)}\Lambda$, $Q = \frac{(d-2)}{4(d-3)}$.

As argued in the main text, we consider static metrics of the form (3.15). The lagrangian (with $k = 1$) then reduces to the particle Lagrangian

$$L = \frac{1}{g}\left(e\phi'^2 - 4Qe'\phi\phi'\right) - \lambda g e\phi^2. \tag{B.10}$$

The prime denotes derivatives with respect to x. It is obvious that the field g does not contribute to the dynamics - the equation of motion for g is simply an expression of reparameterization invariance in the spatial co-ordinate. In fact, all the g-depenence disappears from the equations of motion if we define a new variable $r = \int_0^x \mathrm{d}x'\, g(x')$. We then get

$$\begin{aligned} -2Q\frac{\ddot{e}}{e} + \frac{\dot{\phi}\dot{e}}{\phi e} + \frac{\ddot{\phi}}{\phi} + \lambda &= 0 \\ \frac{\ddot{\phi}}{\phi} + \left(1 + \frac{1}{4Q}\right)\left(\frac{\dot{\phi}}{\phi}\right)^2 - \frac{\lambda}{4Q} &= 0 \\ \frac{\dot{\phi}}{\phi}\left(\frac{\dot{\phi}}{\phi} - 4Q\frac{\dot{e}}{e}\right) + \lambda &= 0, \end{aligned} \tag{B.11}$$

and the dots denote derivatives with respect to r.

Substituting this solution one has for the curvature

$$R[g] = -2a^2\left[1 + \frac{3\gamma}{4Q} + \frac{\gamma(\gamma - 4Q)}{16Q^2}\left(\frac{A - Be^{-2ar}}{A + Be^{-2ar}}\right)^2\right] \tag{B.12}$$

where

$$a^2 = \frac{\lambda}{4Q\gamma}. \tag{B.13}$$

We see from (B.12) that among various solutions we also have the case in which the curvature is constant if either $A = 0$ or $B = 0$.

Since both cases differ only by a co-ordinate transformation, we choose $B = 0$. The longitudinal metric $g_{\alpha\beta}$ is then

$$\mathrm{d}s^2 = -(aCA^q)^2 e^{2aqr}\mathrm{d}t^2 + \mathrm{d}r^2. \tag{B.14}$$

where

$$q = 1 + \frac{\gamma}{4Q} \tag{B.15}$$

This is indeed the AdS_2 metric with the proper warp factor growing linearly in the radial co-ordinate. Our co-ordinates, however, do not cover the whole of AdS. One finds global co-ordinates by defining

$$e^{aqr} = \cos\rho, \tag{B.16}$$

where $0 \leq \rho \leq \pi/2$.

The curvature $R[g] = -\frac{2\ddot{e}}{e}$ obviously simplifies and becomes

$$R[g] = -\frac{\lambda(4Q+\gamma)}{8Q^2\gamma} \tag{B.17}$$

Furthermore, those solutions with A and B non-zero will be analogous in structure to AdS_2/Schwarzschild geometries, though the metric will have a different functional form due to the presence of the non-trivial scalar field.

In $d = 3$ there are small modifications due to the appearence of several $(d-3)$ factors in the general solutions. We can easily proceed here as follows. The one-dimensional form of the action is:

$$L = \frac{e'\phi^{2\prime}}{g} - \Lambda e g \phi^2, \tag{B.18}$$

and so the equations of motion reduce to

$$\begin{aligned} \ddot{\phi}^2 + \Lambda\phi^2 &= 0 \\ \ddot{e} + \Lambda e &= 0 \\ \frac{\dot{\phi}^2\dot{e}}{\phi^2 e} + \Lambda &= 0, \end{aligned} \tag{B.19}$$

after reabsorbing the non-dynamical field g in the definition of the parameter r, as before.

Global structure of the solutions

The metric

$$\mathrm{d}s^2 = -e(r)^2\mathrm{d}t^2 + \mathrm{d}r^2 \tag{B.20}$$

has a horizon when $e(r) = 0$. There are two possible locations of this horizon, depending on the relative sign of the initial conditions A and B.

For $B/A > 0$, $e(r)$ has a simple zero. With the following rescalings of the co-ordinates,

$$
\begin{aligned}
r &= \sqrt{\frac{Q\gamma}{\lambda}} \log B/A + \eta \\
t &= \frac{4Q\gamma}{C\lambda}(4AB)^{-\gamma/8Q-1/2}\tau
\end{aligned}
\tag{B.21}
$$

the metric near the horizon is simply the Rindler space metric,

$$
\mathrm{d}s^2 = -\eta^2 \mathrm{d}\tau^2 + \mathrm{d}\eta^2, \tag{B.22}
$$

and so locally the space is flat.

For $B/A < 0$, we rescale the co-ordinates as follows:

$$
\begin{aligned}
r &= \sqrt{\frac{Q\gamma}{\lambda}} \log |B/A| + \eta \\
t &= (\frac{\lambda |AB|}{Q\gamma})^{-\gamma/8Q-1/2}\tau,
\end{aligned}
\tag{B.23}
$$

and we find the metric

$$
\mathrm{d}s^2 = -\eta^{\gamma/2Q} \mathrm{d}\tau^2 + \mathrm{d}\eta^2 \tag{B.24}
$$

with curvature $R = -\frac{\gamma(\gamma-4Q)}{8Q^2\eta^2}$.