Appendix C

Einstein Spaces and the Holographic Stress-Tensor

C.1 Asymptotic solution of Einstein's equations

In this appendix we collect the results for the solution of the equations (4.11) up to the order we are interested in.

From the first equation in (4.11) one determines the coefficients $g_{(n)}$, $n \neq d$, in terms of $g_{(0)}$. For our purpose we only need $g_{(2)}$ and $g_{(4)}$. There are given by

$$\begin{aligned}
g_{(2)ij} &= \frac{1}{d-2}\left(R_{ij} - \frac{1}{2(d-1)} R\, g_{(0)ij}\right), \\
g_{(4)ij} &= \frac{1}{d-4}\left(-\frac{1}{8(d-1)} D_i D_j R + \frac{1}{4(d-2)} D_k D^k R_{ij}\right. \\
&\quad - \frac{1}{8(d-1)(d-2)} D_k D^k R g_{(0)ij} - \frac{1}{2(d-2)} R^{kl} R_{ikjl} \\
&\quad + \frac{d-4}{2(d-2)^2} R_i{}^k R_{kj} + \frac{1}{(d-1)(d-2)^2} R R_{ij} \\
&\quad \left. + \frac{1}{4(d-2)^2} R^{kl} R_{kl} g_{(0)ij} - \frac{3d}{16(d-1)^2(d-2)^2} R^2 g_{(0)ij}\right).
\end{aligned} \tag{C.1}$$

All curvature expressions and covariant derivatives here are evaluated in the metric $g_{(0)}$. Thus, the above coefficients $g_{(n)}$ are functions of $g_{(0)}$ through the Riemann tensor and its derivatives. The expressions for $g_{(n)}$ are singular when $n = d$. One can obtain the trace and the divergence of $g_{(n)}$ for any n from the last two equations in (4.11). Explicitly,

$$\begin{aligned}
&\mathrm{Tr}\ g_{(4)} = \frac{1}{4}\mathrm{Tr}\ g_{(2)}^2, \qquad \mathrm{Tr}\ g_{(6)} = \frac{2}{3}\mathrm{Tr}\ g_{(2)} g_{(4)} - \frac{1}{6}\mathrm{Tr}\ g_{(2)}^3, \\
&\mathrm{Tr}\ g_{(3)} = 0, \qquad \mathrm{Tr}\ g_{(5)} = 0,
\end{aligned} \tag{C.2}$$

and

$$\nabla^i g_{(2)ij} = \nabla^i A_{(2)ij}, \qquad \nabla^i g_{(3)ij} = 0, \qquad \nabla^i g_{(4)ij} = \nabla^i A_{(4)ij}$$

$$\nabla^i g_{(5)ij} = 0, \qquad \nabla^i g_{(6)ij} = \nabla^i A_{(6)ij} + \frac{1}{6}\text{Tr}\,(g_{(4)}\nabla_j g_{(2)})\,, \tag{C.3}$$

where

$$\begin{aligned}
A_{(2)ij} &= g_{(0)ij}\text{Tr}\,g_{(2)}, \qquad\qquad (\text{C.4})\\
A_{(4)ij} &= -\frac{1}{8}[\text{Tr}\,g_{(2)}^2 - (\text{Tr}\,g_{(2)})^2]\,g_{(0)ij} + \frac{1}{2}(g_{(2)}^2)_{ij} - \frac{1}{4}\,g_{(2)ij}\,\text{Tr}\,g_{(2)},\\
A_{(6)ij} &= \frac{1}{3}\bigg(2(g_{(2)}g_{(4)})_{ij} + (g_{(4)}g_{(2)})_{ij} - (g_{(2)}^3)_{ij} + \frac{1}{8}\,[\text{Tr}\,g_{(2)}^2 - (\text{Tr}\,g_{(2)})^2]\,g_{(2)ij}\\
&- \text{Tr}\,g_{(2)}\,[g_{(4)ij} - \frac{1}{2}(g_{(2)}^2)_{ij}]\\
&- [\frac{1}{8}\text{Tr}\,g_{(2)}^2\text{Tr}\,g_{(2)} - \frac{1}{24}(\text{Tr}\,g_{(2)})^3 - \frac{1}{6}\text{Tr}\,g_{(2)}^3 + \frac{1}{2}\text{Tr}\,(g_{(2)}g_{(4)})]\,g_{(0)ij}\bigg)\,.
\end{aligned}$$

For even $n = d$ the first equation in (4.11) determines the coefficients $h_{(d)}$. They are given by

$$\begin{aligned}
h_{(2)ij} &= 0, \qquad\qquad (\text{C.5})\\
h_{(4)ij} &= \frac{1}{2}g_{(2)ij}^2 - \frac{1}{8}g_{(0)ij}\text{Tr}\,g_{(2)}^2 + \frac{1}{8}(\nabla^k\nabla_i g_{(2)jk} + \nabla^k\nabla_j g_{(2)ik} - \nabla^2 g_{(2)ij} - \nabla_i\nabla_j\text{Tr}\,g_{(2)})\qquad(\text{C.6})\\
&= \frac{1}{8}R_{ikjl}R^{kl} + \frac{1}{48}\nabla_i\nabla_j R - \frac{1}{16}\nabla^2 R_{ij} - \frac{1}{24}RR_{ij} + (\frac{1}{96}\nabla^2 R + \frac{1}{96}R^2 - \frac{1}{32}R_{kl}R^{kl})g_{(0)ij},\\
h_{(6)ij} &= \frac{2}{3}(g_{(4)}g_{(2)} + g_{(2)}g_{(4)})_{ij} - \frac{1}{3}g_{(2)ij}^3 - \frac{1}{6}g_{(4)ij}\text{Tr}\,g_{(2)}\\
&+ \frac{1}{6}g_{(0)ij}(3\text{Tr}\,g_{(6)} - 3\text{Tr}\,g_{(2)}g_{(4)} + \text{Tr}\,g_{(2)}^3)\\
&-\frac{1}{12}[-\frac{1}{4}\nabla_i\nabla_j\text{Tr}\,g_{(2)}^2 - \nabla^k\nabla_i g_{(4)jk} - \nabla^k\nabla_j g_{(4)ik} + \nabla^2 g_{(4)ij}\\
&+g_{(2)}^{kl}[\nabla_l\nabla_i g_{(2)jk} + \nabla_l\nabla_j g_{(2)ik} - \nabla_l\nabla_k g_{(2)ij}]\\
&+\frac{1}{2}\nabla^k\text{Tr}\,g_{(2)}(\nabla_i g_{(2)jk} + \nabla_j g_{(2)ik} - \nabla_k g_{(2)ij})\\
&+\frac{1}{2}\nabla_i g_{(2)kl}\nabla_j g_{(2)}^{kl} + \nabla_k g_{(2)il}\nabla^l g_{(2)j}{}^k - \nabla_k g_{(2)il}\nabla^k g_{(2)j}{}^l]. \qquad\qquad (\text{C.7})
\end{aligned}$$

C.2 Divergences in terms of the induced metric

In this appendix we rewrite the divergent terms of the regularised action in terms of the induced metric at $\rho = \epsilon$. This is needed in order to derive the contribution of the counter-terms to the stress-energy tensor.

The coefficients $a_{(n)}$ of the divergent terms in the regulated action (4.25) are given by

$$\begin{aligned}
a_{(0)} &= 2(1-d),\\
a_{(2)} &= b_{(2)}(d)\,\text{Tr}\,g_{(2)},\\
a_{(4)} &= b_{(4)}(d)\,[(\text{Tr}\,g_{(2)})^2 - \text{Tr}\,g_{(2)}^2],
\end{aligned}$$

$$a_{(6)} = \left(\frac{1}{8}\,\mathrm{Tr}\ g_{(2)}^3 - \frac{3}{8}\,\mathrm{Tr}\ g_{(2)}\mathrm{Tr}\ g_{(2)}^2 + \frac{1}{2}\,\mathrm{Tr}\ g_{(2)}^3 - \mathrm{Tr}\ g_{(2)}g_{(4)}\right), \qquad \text{(C.8)}$$

where $a_{(6)}$ is only valid in six dimensions and the numerical coefficients in $a_{(2)}$ and $a_{(4)}$ are given by

$$\begin{aligned} b_{(2)}(d \neq 2) &= -\frac{(d-4)(d-1)}{d-2}, \\ b_{(2)}(d = 2) &= 1, \\ b_{(4)}(d \neq 4) &= \frac{-d^2+9d-16}{4(d-4)}, \\ b_{(4)}(d = 4) &= \frac{1}{2}. \end{aligned} \qquad \text{(C.9)}$$

Notice that the coefficients $a_{(n)}$ are proportional to the expression for the conformal anomaly (in terms of $g_{(n)}$) in dimension $d = n$ [70].

The counter-terms can be rewritten in terms of the induced metric by inverting the relation between γ and $g_{(0)}$ perturbatively in ϵ. One finds

$$\begin{aligned} \sqrt{g_{(0)}} &= \epsilon^{d/2}\left(1 - \frac{1}{2}\,\epsilon\,\mathrm{Tr}\ g_{(0)}^{-1}g_{(2)} + \frac{1}{8}\,\epsilon^2\,[(\mathrm{Tr}\ g_{(0)}^{-1}g_{(2)})^2 + \mathrm{Tr}\ (g_{(0)}^{-1}g_{(2)})^2] + \mathcal{O}(\epsilon^3)\right)\sqrt{\gamma}, \\ \mathrm{Tr}\ g_{(2)} &= \frac{1}{2(d-1)}\frac{1}{\epsilon}\left(R[\gamma] + \frac{1}{d-2}(R_{ij}[\gamma]R^{ij}[\gamma] - \frac{1}{2(d-1)}R^2[\gamma]) + \mathcal{O}(R[\gamma]^3)\right), \\ \mathrm{Tr}\ g_{(2)}^2 &= \frac{1}{\epsilon^2}\frac{1}{(d-2)^2}\left(R_{ij}[\gamma]R^{ij}[\gamma] + \frac{-3d+4}{4(d-1)^2}R^2[\gamma] + \mathcal{O}(R[\gamma]^3)\right). \end{aligned} \qquad \text{(C.10)}$$

The terms cubic in curvatures in (C.10) give vanishing contribution in (4.27) up to six dimensions.

Putting everything together we obtain that the counter-terms, rewritten in terms of the induced metric, are given by

$$\begin{aligned} S^{\mathrm{ct}} &= -\frac{1}{16\pi G_{\mathrm{N}}}\int_{\rho=\epsilon}\sqrt{\gamma}\,\Big[2(1-d) + \frac{1}{d-2}R \\ &- \frac{1}{(d-4)(d-2)^2}(R_{ij}R^{ij} - \frac{d}{4(d-1)}R^2) - \log\epsilon\, a_{(d)} + \ldots\Big], \end{aligned} \qquad \text{(C.11)}$$

where all quantities are now in terms of the induced metric, including the one in the logarithmic divergence. These are exactly the counter-terms in [11, 43, 83] except that these authors did not include the logarithmic divergence. Equation (C.11) should be understood as containing only divergent counter-terms in each dimension. This means that in even dimension $d = 2k$ one should include only the first k counter-terms and the logarithmic one. In odd $d = 2k+1$, only the first $k+1$ counter-terms should be included. The logarithmic counter-terms appear only for d even. The counter-terms in (C.11) render the renormalised action finite up to $d = 6$. This covers all cases relevant for the AdS/CFT correspondence. It is straightforward but tedious to compute the necessary counter-terms for $d > 6$. From (C.11) one straightforwardly obtains (4.30).

C.3 Relation between $h_{(d)}$ and the conformal anomaly $a_{(d)}$

We show in this appendix that the tensor $h_{(d)}$ appearing in the expansion of the metric in (4.9) when d is even is a multiple of the stress tensor derived from the action $\int a_{(d)}$. ($a_{(d)}$ is, up to a constant, the holographic conformal anomaly).

This can be shown by deriving the stress-energy tensor of the regulated theory at $\rho = \epsilon$ in two ways and then comparing the results. In the first derivation one starts from (4.24) and obtains the regulated stress-energy tensor as in (4.29). Expanding $T_{ij}^{\rm reg}[\gamma]$ in ϵ (keeping $g_{(0)}$ fixed) we find that there is a logarithmic divergence,

$$T_{ij}^{\rm reg}[\gamma;\log] = \frac{1}{8\pi G_{\rm N}} \log\epsilon \; (\frac{3}{2}d - 1) h_{(d)ij}. \tag{C.12}$$

On the other hand, one can derive $T_{ij}^{\rm reg}[\gamma]$ starting from (4.25). One has to first rewrite the terms in (4.25) in terms of the induced metric. This is done in the previous appendix. Once $T_{ij}^{\rm reg}[\gamma]$ has been derived, we expand in ϵ. We find the following logarithmic divergence:

$$T_{ij}^{\rm reg}[\gamma;\log] = \frac{1}{8\pi G_{\rm N}} \log\epsilon \left((1-d) h_{(d)ij} - T_{ij}^{a},\right), \tag{C.13}$$

where T_{ij}^{a} is the stress-energy tensor of the action $\int {\rm d}^d x \sqrt{\det g_{(0)}}\, a_{(d)}$. If follows that

$$h_{(d)ij} = -\frac{2}{d} T_{ij}^{a}. \tag{C.14}$$

We have also explicitly verified this relation by brute-force computation in $d = 4$.

C.4 Asymptotic solution of the scalar field equation

We give here the first two orders of the solution of the equation (4.70)

$$\begin{aligned}
\phi_{(2)} &= \frac{1}{2(2\Delta - d - 2)} \left(\Box_0 \phi_{(0)} + (d-\Delta)\phi_{(0)} {\rm Tr}\ g_{(2)}\right), \\
\phi_{(4)} &= \frac{1}{4(2\Delta - d - 4)} \Bigg(\Box_0 \phi_{(2)} - 2\,{\rm Tr}\ g_{(2)}\phi_{(2)} - \frac{1}{2}(d-\Delta)\left[{\rm Tr}\ g_{(2)}^2\, \phi_{(0)} - 2{\rm Tr}\ g_{(2)}\, \phi_{(2)}\right] \\
&\qquad - \frac{1}{\sqrt{g_{(0)}}}\,\partial_\mu(\sqrt{g_{(0)}}\, g_{(2)}^{\mu\nu}\partial_\nu \phi_{(0)}) + \frac{1}{2}\partial^i {\rm Tr}\ g_{(2)} \partial_j \phi_{(0)}\Bigg),
\end{aligned} \tag{C.15}$$

where in $\Box_0$ the covariant derivatives are with respect to $g_{(0)}$.

If $2\Delta - d - 2k = 0$ one needs to introduce a logarithmic term in order for the equations to have a solution, as discussed in the main text. For instance, when $\Delta = \frac{1}{2}d + 1$, $\phi_{(2)}$ is undetermined, but instead one obtains for the coefficient of the logarithmic term,

$$\psi_{(2)} = -\frac{1}{4}\left(\Box_0 \phi_{(0)} + (\frac{d}{2} - 1)\,\phi_{(0)} {\rm Tr}\ g_{(2)}\right). \tag{C.16}$$