Chapter 1

Introduction

Ever since in 1974 Hawking discovered that black holes emit radiation [65], there has been great controversy about the fact that black holes can evaporate, and about their fate after have they have done so. Indeed, as is well-known, pairs of particles and anti-particles can form in vacuum. These particles however tend to recombine and, under usual circumstances, they will annihilate each other after a very short time. However, when these pairs form in the vicinity of a black hole, there is a small chance for the particle to have just enough energy to escape to infinity, whereas its partner with negative energy is doomed to fall into the black hole. Obviously this is a small effect, as the probability for a Hawking particle to have enough energy to escape to infinity, where we can measure it, is extremely small. Yet the mere idea that such a process is possible is a great challenge for theoretical physics, for it raises the question what would happen if we were able to isolate a black hole (in our minds) so that nothing falls in but it only can emit particles and hence evaporate. In fact, small black holes will evaporate very fast, as the temperature of Hawking radiation is inversely proportional to the mass:

$$T = \frac{\hbar c^3}{8\pi k G_{\rm N} M} = 6 \cdot 10^{-8} (M_\odot / M)\,{\rm K}, \tag{1.1}$$

where $M_\odot$ is the solar mass. For a black hole as heavy as the sun this is a very tiny effect, but small black holes, like the ones that were formed in the early universe, may have a mass small enough to emit strong radiation.

The controversy we alluded to above is not so much concerned with the fact that black holes emit radiation, but rather with the nature of the radiation: it is purely thermal. Its spectrum is that of black-body radiation, which means that it contains little information about the initial state of the black hole. Take for example a page of this thesis and burn it (this is just a thought experiment). After the paper is completely burned, all the precious information that was in it is lost. A close look at the few ashes that are left behind or an analysis of the radiation that is emitted will not help us puzzling out what was written on the paper. We can recover a great deal of the information about it — its chemical composition, etc.—, but not the detailed information about how molecules were precisely arranged on the surface of the sheet.

With black holes the situation is very similar. The Hawking radiation that is emitted is coarse-grained, it does not contain precise information about for example how the black hole exactly formed and all its past history. This is why information is lost in the process of evaporation.

Yet for the burned page we know this is not completely true: if we were able to keep track of each single molecule after the page is burned, applying the laws of physics (and chemistry) we would be able to give their precise configuration when the thesis was still intact, and so we would succeed at recovering the lost information. This is nothing else than the statement that thermodynamics can be derived from microscopic physics by a coarse-graining procedure. In quantum mechanical terms, if the final state is pure, the initial state must be pure as well unless there is a violation of quantum mechanics. Now if Hawking's argument is correct, black holes violate quantum mechanics, as their final state is mixed and not pure.

One can hardly overestimate the importance of Hawking's paradox for our understanding of nature. If true, it points to a fundamental discrepancy between general relativity and quantum mechanics, and so it is extremely important to find out whether there is a mistake in the formulation of Hawking's argument, or whether we have to change the fundamental laws of physics by allowing quantum mechanics to be even less "classical". Indeed, conventional quantum mechanics already leads to conceptual difficulties, but a theory where transitions between pure and mixed states are allowed would be even less transparent and, what is worse, it would be very unlikely to respect basic principles of physics like energy conservation.

A few years before Hawking radiation was discovered, Jacob Bekenstein developed the laws of black hole thermodynamics, based on the analogy between black hole mechanics and thermodynamics [19]. He argued that, up to a constant, the entropy of a black hole must be proportional to its area:

$$S_{\rm BH} = \frac{kc^3 A}{4G\hbar}, \tag{1.2}$$

and the precise proportionality factor of 1/4 was only determined when Hawking radiation was discovered. Based on the analogy with statistical mechanics, this suggests that the area of the black hole is a measure for the number of microscopical states that give rise to the same macroscopic black hole of mass M, charge Q and angular momentum J.

In string theory, several microscopic countings have been made that confirm the area-law (1.2) with the right proportionality coefficient. Though performed for so-called extremal and near-extremal black holes, which are presumably not of much astrophysical relevance, these countings give, for the first time, a microscopical explanation of the black hole entropy formula.

Motivated by the above relation between entropy and area, in 1993 't Hooft conjectured that at Planckian energies our world is not three-, but two-dimensional [112]. The argument, in simplified form, was as follows. Consider a closed region of space-time of volume $V \sim R^3$ and energy E and ask how many physical states there are in this region. For the states to be physical and thus measurable for an outside observer, we must require that the radial size of the region we consider is larger than the size of its Schwarzschild radius. Otherwise the surface would lie within its own horizon and would be hidden to the observer outside. Since the Schwarzschild radius is given by the energy inside, we get the bound

$$2E < R \tag{1.3}$$

i.e. the Schwarzschild radius should always be smaller than the actual radius, and so the energy density inside the volume is not allowed to be too large.

Given ordinary quantum field theory, the most probable state would be a gas at some temperature T. Its energy would be given by Boltzmann's law,

$$E \sim VT^4. \tag{1.4}$$

In what follows we suppress all multiplicative constants of order 1. The total entropy is

$$S \sim VT^3, \tag{1.5}$$

and so combining (1.3) with (1.4) one gets a bound on the temperature. This gives, for the entropy,

$$S < V^{\frac{1}{2}} \sim A^{\frac{3}{4}}, \tag{1.6}$$

which for large area does not exceed the entropy of a black hole of the same size. Thus, black holes have the largest entropy ordinary matter can possibly have. In fact, they have a larger entropy than what is suggested by the stronger bound (1.6). This is not surprising, as any form of matter will form a black hole if we increase its energy density more and more.

What is surprising is that the limit on the entropy is set by the area, (1.2), and not by the volume. 't Hooft's explanation was that most of the states of field theory are not physical, for their energy is so large that they are confined inside their own Schwarzschild radius. So, the expectation is that gravitational physics reduces the number of physical degrees of freedom: states with energy corresponding to a Schwarzschild size larger than the size of the physical system are not physical and so should be disregarded, hence the number of states grows exponentially with the area instead of the volume. It was then conjectured that quantum gravity should be described by a topological field theory, in the sense that all its degrees of freedom live on the boundary. This is called the *holographic hypothesis*.

There have been various generalisations of the holographic principle which we will not go in detail into, as in this thesis we will only consider the, from a geometrical point of view, most simple cases. In general, one has to define the boundary of a certain region, and its inside and outside. This can be done by looking at the propagation of light rays from a certain region [22].

Much progress in the understanding of the holographic principle came from very different considerations when in 1997 Maldacena conjectured the so-called AdS/CFT correspondence [87]. The AdS/CFT correspondence goes back to the long-ago conjectured relationship between gauge theories and strings [118]. It relates string and gravity theories in a certain back-ground (so-called "anti-de Sitter", AdS for short) to certain field theories which do not contain gravity (CFT stands for "conformal field theory"). AdS space is a space with a timelike boundary, and in this sense it can be compared with a "box" (one can think of it as a cylinder of circular base). The field theory is defined at the boundary of the space, which corresponds to the wall of the cylinder. Thus, the field theory lives in a space of one dimension less. The AdS/CFT correspondence thus gives a simple realisation of the holographic principle: the gravitational degrees of freedom in the bulk can be arranged in such a way that they describe a non-gravitational theory living on the boundary of the space.

The holographic principle is not only a statement about the number of microstates of the theory. It also implicitly assumes that these degrees of freedom reorganise on the boundary in a somehow physically meaningful way. This implies that the boundary theory should at least respect causality. The AdS/CFT correspondence is a nice arena to perform tests of causality, and in fact some non-trivial tests have been performed with black holes and collisions between massless particles. Although some bizarre behaviour has been found [98, 108, 85] from the boundary point of view, so far no contradictions have been perceived with the causality principles of quantum field theory. Perhaps even more surprising than the fact that the theory lives on the boundary is the fact that the AdS/CFT correspondence relates bulk gravity to one of the field theories that were already known.

In this thesis we are mainly concerned with two different approaches to holography. The first one is an analysis of the eikonal regime of quantum gravity, where the theory reduces to a topological field theory. This is the regime where particles interact at high energies but with small momentum transfer. We also consider quantum gravity away from the extreme eikonal limit and find indications that the theory remains topological. The second approach we pursue is the AdS/CFT correspondence, where one can ask very precise questions about the way the geometry of the bulk and the matter fields are encoded in the boundary theory. We also study warped compactifications, where our d-dimensional world is regarded as a slice of a $d+1$-dimensional space-time, and analyse in detail the question as to where the d-dimensional observer can find the information about the extra dimension. Much of what we do does not assume string theory directly, although most of our results can be embedded in string theory, and in fact we think string theory is probably the best way to understand and think about our results. In particular, the discussion of the AdS/CFT correspondence does assume string theory. Even though in this thesis we investigate two apparently very different approaches, our aim is in fact to apply them to situations where both can be used. In this way we are naturally led to considering Planckian scattering in AdS. This will be studied in chapter 3, where we make a few preliminary remarks about the relation between both.

The thesis is organised as follows. The first chapter is introductory: we first explain the sorts of problems related to black holes and Hawking radiation which motivate this work. We explain why the assumption of the holographic principle can be a way to solve them. Then we review the features of quantum gravity in the eikonal regime, string theory and the AdS/CFT correspondence. Particular emphasis is laid on how holography arises in the context of quantum gravity and of the AdS/CFT correspondence. In chapter 2 we study in detail high-energy scattering between massless particles: classical and quantum mechanical features of gravitational scattering, and how to go beyond the eikonal approximation. In chapter 3 we generalise some of these results to spaces with a cosmological constant (positive or negative) and find the corresponding dual theories. A particularly interesting case is that of a positive cosmological constant. We believe our results are relevant to the discussions in [22, 67, 45] on the possibility of describing holographic duals of de Sitter space. In chapter 4 we study the reconstruction of space-time and of space-time fields from the CFT. We do this perturbatively in the distance to the boundary. We develop a systematic method to regularise and renormalise the bulk action, and interpret our results from the CFT point of view. In chapter 5 we reinterpret the counter-terms of the gravitational action as generating the dynamics from the point of view of an observer living on a brane of codimension 1. We analyse the cases of asymptotically AdS, dS and flat space-time.

1.1 Holography in Quantum Gravity

The most clear and astonishing example of a holographic map between a gravitational and a non-gravitational theory is perhaps the AdS/CFT correspondence. It remains very mysterious, however, how holography may work if the bulk space-time is not AdS but asymptotically flat. In particular, a satisfactory description of the four-dimensional Schwarzschild black hole is still lacking.

As a matter of fact there exists a holographic description, if not of an evaporating Schwarzschild black hole, of a Rindler space-based model that is to mimic the most important features of the near-horizon region of the four-dimensional black hole. This is the S-matrix description discussed by 't Hooft [113], which we are going to examine in detail in this thesis. However, even if this is truly a holographic model, the quantum mechanical properties of the model are not well understood beyond the eikonal approximation, and no entropy formula has been derived. Nevertheless, it is quite remarkable that this model does exhibit explicitly how the information that falls into the black hole is stored into the outgoing radiation without violating any no-quantum-copying-machine principle. In particular, one can compute an approximated S-matrix. It furthermore has a striking similarity with string theories and non-commutative geometry. It also gives interesting insights in the non-perturbative regime of quantum gravity in the eikonal approximation. For these reasons, we think that the model is worth studying, the more because it is applicable in the context of AdS where we also have a dual CFT description. It would be extremely interesting if one could "compare" both holographic duals, and we will make a few preliminary remarks in that direction. It is clear that a cross-fertilisation between the S-matrix model, where the issue of unitarity is exhibited explicitly, and the CFT description, for which there exists an extraordinarily precise dictionary, is most desirable (for a discussion of the issue, see, e.g., [85, 106, 97]).

The next sections are an introduction to some aspects of the eikonal regime of quantum gravity, first in the specific context of point-like particles on a fixed background, and later in a more general set-up. We review in particular how holography arises in the context of quantum gravity. There are many other relevant papers on the subject (see, e.g., [73, 74]), but for the purpose of this thesis we restrict ourselves to the ones that will be used in later sections.

1.1.1 Quantum Gravity in the Eikonal Regime

The main ingredient of the S-matrix Ansatz is the gravitational interactions between in-going particles and out-coming radiation on a black hole horizon. These interactions are not taken care of in the derivation of Hawking radiation, and because of the extreme high frequencies of the in-falling modes these interactions cannot be neglected.

If quantum gravity would be non-predictable in the way originally discussed by Hawking, we would have to enlarge the uncertainty in quantum mechanics to allow for an uncertainty in the state of the wave-function: on top of the statistical description of observables postulated by quantum mechanics, there would be an uncertainty in the quantum state [66]. However, there are strong reasons to believe that gravity can be reconciled with quantum mechanics without giving up unitarity. String theory, and in particular the AdS/CFT correspondence, supports such a view. Nevertheless it is important for the understanding of quantum gravity to be able to point to a loophole in the original argument. Although the contents of this section have already been discussed at length in [113] and other publications by 't Hooft, we will review the S-matrix Ansatz once more because it is the starting point for other considerations in the next chapters.

The basic idea is to take into account the fact that in-going and out-coming particles interact gravitationally at the horizon. If the black hole was formed by some in-falling matter configuration, there will be traces of its initial state on the geometry near the horizon, and so, when Hawking radiation is emitted, it will be scattered off that non-trivial surrounding geometry.

Consider a Schwarzschild black hole in a typical state, say a superposition of in-going and out-going particles. States for the Schwarzschild (Rindler) observer are related to the Kruskal (Minkowski) vacuum by the well-known Bogolyubov transformation,

$$a_{\tilde{k},\omega} \quad = \quad \frac{1}{\sqrt{1-e^{-2\pi\omega}}}\left[b_{\tilde{k},\omega}+e^{-\pi\omega}b^{\dagger}_{-\tilde{k},-\omega}\right]. \tag{1.7}$$

The operator a annihilates a particle of energy ω and momentum $\tilde{k}$ in Rindler space, whereas b is directly related to the annihilation operator in Minkowski space. One can easily check that this mixing between creation and annihilation operators gives rise to the following relation between states:

$$|0\rangle_{\rm M}=\prod_{\tilde{k},\omega}\sqrt{1-e^{-2\pi\omega}}\sum_{n=0}^{\infty}e^{-n\pi\omega}|n,n\rangle_{\tilde{k},\omega}. \tag{1.8}$$

Consider now the process of "purifying" such a state by removing first one particle and subsequently all the others, until we are left with the vacuum. For the Kruskal observer, more and more particles are being added to his state, with such tremendous energies that they will interact gravitationally, eventually forming small and even big black holes. It is clear that in such a situation the Bologyubov transformation (1.7) will not be correct, as gravitational interactions were neglected in its derivation. So, for the Kruskal observer, the vacuum of the Rindler observer is not at all a vacuum state nor a thermal bath of particles, but it will rather be a highly complicated, gravitationally interacting state. If we had a way of adding or removing particles from our state, keeping track of correlations with other particles, we could then reach any state in Fock space if only we had one reference state.

To realise this in practise goes beyond present knowledge, but we can give an approximated picture. In the next section we will consider an arbitrary state of out-going particles and add one in-going particle to see how the state changes. Repeating this procedure many times, we can compute the S-matrix of the whole process, up to an unknown phase which is the transition element between those reference in- and out-states. Notice that in this context it is not possible to compute this phase because in the eikonal approximation which we will be considering there is no black hole formation. To describe the creation of small black holes one has to consider the full transfer of momentum. We will not discuss black hole creation, but we will discuss how to go beyond the eikonal approximation. Black hole formation is a very important issue which has been considered in a simplified 2+1-dimensional set-up in [89]. Important related discussions in the context of the AdS/CFT correspondence and string theory can be found in [10, 82].

The natural objects to have falling into a black hole are massless objects, since any massive object that is falling into the black hole will be boosted to the speed of light with a tremendous energy. Therefore, we concentrate on massless point particles. The momenta of in-falling particles grow exponentially with Schwarzschild time, whereas momenta of out-coming particles decrease exponentially. A time lapse $\delta t = 4M\gamma$ in Rindler co-ordinates corresponds to a Lorentz-boost in Kruskal co-ordinates,

$$\begin{aligned} u &\rightarrow e^{\gamma}\, u \\ v &\rightarrow e^{-\gamma}\, v \\ p_u &\rightarrow e^{-\gamma}\, p_u \\ p_v &\rightarrow e^{\gamma}\, p_v, \end{aligned} \tag{1.9}$$

in co-ordinates where the future horizon is at $v = 0$, and the past horizon at $u = 0$. So the momentum of in-falling particles grows exponentially as they approach the horizon.

We anticipate that the gravitational effect of such a massless particle on the trajectories of the out-going Hawking particles takes the form of a shift,

$$u \rightarrow u + p_v^{\text{in}}(\theta', \phi')\, f(\theta, \phi, \theta', \phi'), \tag{1.10}$$

so also the horizon shifts and out-coming particles come out at time $u = p_v^{\text{in}} f$. This means that the size of the black hole has become larger. Notice that, according to (1.9), this shift grows larger and larger as Schwarzschild time goes by, and so at some point it will not be negligible. The point of view we advocate in this thesis is that this is a relevant effect that should be taken into account in the unitarity argument. Indeed, as explained in [113], there seems to be a hidden assumption in the derivation of the Hawking spectrum. This derivation performs a co-ordinate transformation from Minkowski to Rindler co-ordinates in the asymptotic region, where the energy of particles is rather low, and so this transformation seems a good approximation, at least as long as one computes macroscopic properties like the intensity of the emitted flux. However, when it comes to microscopic correlations between the radiation and the in-going particles, this approximation fails because it does not take into account the fact that particles collided at very high energy near the horizon and so they remain correlated afterwards.

In this thesis we will concentrate on the effect of the shift (1.10). Since the shift only takes into account the in-going momentum and not other possible charges of the particle, this is only an approximation to the real problem. However, notice that in a world with no other charges momentum would be enough to recover the information about the particle that was sent in. In realistic models this is also a good approximation because at those energies gravity is the dominant interaction. Electromagnetic interactions are subdominant and they can be easily incorporated in this model, but other charges are more difficult to account for. For a discussion of this issue we refer to [113, 76].

Several objections have been raised against the existence of an S-matrix with such properties. The strongest one seems to be the no-quantum-copying-machine principle [107], which can be formulated as follows. Imagine sending some pure state into a black hole, and assume there is some linear operator X copying this information on an outgoing state. Since the Hilbert space decomposes into an in- and an out-component, $\mathcal{H} = \mathcal{H}_{\rm in} \otimes \mathcal{H}_{\rm out}$, the operator X acts as

$$X\left(|\psi\rangle_{\rm in} \otimes |\phi\rangle_{\rm out}\right) = |\psi\rangle_{\rm in} \otimes |\psi\rangle_{\rm out}. \tag{1.11}$$

However, by letting X act on a superposition $|\psi\rangle_{\rm in} = |\alpha\rangle_{\rm in} + |\beta\rangle_{\rm in}$ one easily sees that an operator defined as above would not be linear and so would violate one of the basic principles of quantum mechanics, the linear evolution of states. Therefore, according to this argument there is no such a thing as a quantum copying machine.

This argument assumes that Hilbert space can be separated into an in- and an out-component, but that turns out not to be true in the S-matrix Ansatz. Actually both Hilbert spaces are complementary just like position and momentum space are in quantum mechanics. So we are forced to describe physics in either one Hilbert space or in the other, but not in both at the same time. The operator that will do the job of "copying" the information of the in-going waves to out-coming radiation will be $X = e^{ip_{\rm in} f p_{\rm out}}$, where p is the momentum of the in- or out-going waves, and f is a function of the impact parameter. This operator certainly acts linearly on wave-functions, and it is actually directly related to the S-matrix. The fact that the in- and out-Hilbert spaces are complementary means that we cannot do measurements on outgoing waves without influencing the outcomes of measurements done on in-going waves: if we choose to measure certain observables outside the black hole, this will imply an uncertainty for the outcomes of measurements inside the black hole.

It is easy to see how the shift (1.10) comes about. An in-falling massless particle with momentum $p_{\rm in}$ is described by the following shock-wave metric [2]

$$\mathrm{d}s^2 = 2\mathrm{d}u(\mathrm{d}v - p_{\rm in}\delta(u)\, f\, \mathrm{d}u) + \mathrm{d}x^2 + \mathrm{d}y^2. \tag{1.12}$$

The geodesics of massless test particles in this metric are easy to compute and give

$$\begin{aligned} v(u) &= v_0 + p_{\rm in}\,\theta(u)\left(f + u\,\frac{\partial x^i}{\partial u}\,\partial_i f\right) \\ x^i(u) &= x^i_0 - \frac{1}{2}\,p_{\rm in}\,\partial_i f\, u\,\theta(u) \end{aligned} \tag{1.13}$$

where u parametrises the null geodesic. The function f is given by

$$f = -4G_{\rm N}\,\log(x^2 + y^2). \tag{1.14}$$

The impact parameter is the transverse distance between both particles, $b = \sqrt{x^2 + y^2}$ (in co-ordinates where one of the particles is at the origin). Therefore, for large transverse separations as compared to the Planck length, the derivatives of f, $\partial_i f \sim \frac{1}{b}$, can be neglected. More precisely, we have the following small dimensionless parameter: $\varepsilon = G_{\rm N} p_{\rm in}/b$. The approximation where this parameter is taken to be small is called the eikonal approximation. In that approximation, we see from the above formulae that v is modified purely by a shift as the test particle crosses the world-line of the in-going particle, whereas the transverse co-ordinates remain unchanged. So, after the collision the particle continues along the same straight line, and the only effect of the collision is a time delay. This in turn means that the momentum transfer during the collision is negligible.

Next we briefly summarise the considerations leading to the black hole S-matrix [113]. Take some reference state $|p_{\rm in}\rangle$ of particles falling into a black hole, distributed over the horizon as $p_{\rm in} = p_{\rm in}(\Omega)$. Then assume that we have an element of the S-matrix that describes the formation and evaporation of the black hole,

$$\mathcal{N} = \langle {\rm in}_0|{\rm out}_0\rangle = \langle p_{{\rm in},0}(\Omega)|p_{{\rm out},0}(\Omega)\rangle. \tag{1.15}$$

If we perturb the in-going state by adding some momentum $\delta p_{\rm in}$, $p_{\rm in} \to p_{\rm in} + \delta p_{\rm in}$, out-going particles will be shifted according to (1.13):

$$\delta v = f \delta p_{\rm in}. \tag{1.16}$$

So the out-state is modified by:

$$|p'_{\rm out}\rangle = e^{i\delta v\, \hat{p}_{\rm out}}|p_{{\rm out},0}\rangle, \tag{1.17}$$

the caret meaning that we are generating a shift. So we get a new S-matrix element

$$\langle p'_{\rm out}|p_{\rm in}\rangle = \mathcal{N}\, e^{-i\delta p_{\rm in}\, p_{\rm out} f}. \tag{1.18}$$

This way we can reach any state $|p_{\rm out}\rangle$ from a known state $|{\rm in}_0\rangle$ by the successive addition of infinitesimal amounts of momentum, and so we get

$$\langle p_{\rm out}|p_{\rm in}\rangle = \mathcal{N}' e^{-ip_{\rm in}p_{\rm out} f} \tag{1.19}$$

where we filled in the expression for the shift. The magnitude of $\mathcal{N}'$ is fixed by unitarity, but its phase is arbitrary and may depend on the details of the formation of the black hole. We refer to [113] for further details.

When computing the scattering amplitude from (1.19), one finds [114] the Veneziano amplitude for scattering between strings, with an imaginary string constant related to Newton's constant.

A Fourier transform of the above gives

$$\begin{aligned} \langle p_{\rm out}(\Omega)|p_{\rm in}(\Omega)\rangle \;&=\; \int \mathcal{D}u_{\rm in}\mathcal{D}u_{\rm out}\; \exp\Big[\, i\int {\rm d}^2\Omega\,(\partial u_{\rm in}\partial u_{\rm out}+ \\ &\qquad + p_{\rm in}u_{\rm in} - p_{\rm out}u_{\rm out} + u_{\rm in}u_{\rm out})\Big] \end{aligned} \tag{1.20}$$

which resembles very much the path integral over the world-sheet action of a string. Notice that there is a mass term that breaks conformal invariance. This term, however, is absent if instead of a black hole we consider a Minkowski background[1].

The fields $u_{\rm in}$ and $u_{\rm out}$ are introduced as the Fourier transforms of $p_{\rm in}$ and $p_{\rm out}$, and so at the quantum level we have

$$\begin{aligned} [u_{\rm in}(\Omega), p_{\rm in}(\Omega')] &= i\delta(\Omega - \Omega') \\ [u_{\rm in}(\Omega), p_{\rm in}(\Omega')] &= i\delta(\Omega - \Omega') \\ [u_{\rm in}(\Omega), u_{\rm out}(\Omega')] &= if(\Omega - \Omega'). \end{aligned} \tag{1.21}$$

We see that gravity drastically changes the structure of space-time as seen by massless particles. Co-ordinates between particles become mutually non-commuting operators.

This has far-reaching consequences for the interpretation of Minkowski space as the near-horizon region of Kruskal space. The positions of particles that fall into a black hole are correlated with the positions of the emitted particles, and so Hilbert space does not reduce to a direct product of in and out Hilbert spaces. In other words, modifying the state of in-falling particles does modify the state of the Hawking radiation that is sent out. This obviously reduces the dimensionality of Hilbert space drastically, although we haste to add that every state u still depends on a continuous variable, the angular variable Ω, and so a transverse cutoff is still needed in this crude approximation.

It should now be clear why it is claimed that high-energy scattering presents holographic features. The theory that one gets is the sigma model (1.20), whose fields are defined on a two-dimensional surface, the two-sphere for the case of a Kruskal background. This can be best understood in the context of the results of [121], which we will review in the next section.

1.1.2 Quantum Gravity as a Topological Field Theory

In the previous section we saw that collisions of massless particles at high energies exhibit great similarity with strings, the reason being the extended nature of the gravitational shock-wave. One can wonder whether this is a specific feature of the shock-wave solution, or a general property of gravity at high energies.

In references [73, 74, 121] it was shown that most of the features of the S-matrix model can be understood as specific properties of the eikonal limit of quantum gravity. Indeed, in this regime quantum gravity can be shown to have zero bulk degrees of freedom, all the degrees of freedom living purely on the boundary. So in that regime quantum gravity reduces to a topological field theory. The boundary here is the usual asymptotic null boundary of Minkowski space if we are talking about asymptotically flat spaces, but in chapter 3 we will see that it can also be the boundary of dS and AdS space. This result is at first extremely puzzling, as in general one would expect gravity to have a nonzero number of degrees of freedom in the bulk.

The derivation by Verlinde and Verlinde also sheds light on the validity regime of the S-matrix ansatz. As we will see, the eikonal regime is a perturbative regime as far as transverse processes are concerned, but is non-perturbative in the longitudinal length

[1]When talking about a *background* in the context of shock-wave solutions, we mean a shock-wave on some background space-time.

scale. We will review this argument in some detail here, as it will be the starting point of our generalisation in chapter 3.

The basic argument involves dimensional analysis of the different length scales in the problem. This is a usual argument used in field theory to derive the perturbation expansion. We can set all the dependence on dimensionful quantities into the metric by a rescaling of co-ordinates. Imagine that the typical length scale of the problem is given by some quantity ℓ, then the metric scales like

$$G_{\mu\nu} = \ell^2 \hat{G}_{\mu\nu}, \tag{1.22}$$

where $\hat{G}_{\mu\nu}$ is dimensionless. In four dimensions, the Einstein-Hilbert action scales like

$$S_{\text{EH}}[\ell^2 \hat{G}] = \ell^2 S_{\text{EH}}[\hat{G}]. \tag{1.23}$$

Now although in the path integral one integrates over all metrics, one expects that the dominant contribution will be given by those configurations whose size is that of the physical system, and so it seems reasonable to expect that $\hat{G}_{\mu\nu}$ is typically of order 1. With this assumption, the coupling constant multiplying the action is

$$g = \frac{\ell_{\text{Pl}}}{\ell}. \tag{1.24}$$

This argument is commonly used to argue that when energies are of the Planck size, the theory is strongly coupled and so one needs the full quantum gravity theory to make sensible predictions.

Consider, however, a process where particles collide with Planckian energies but almost head-on. In such a collision, the longitudinal variables $x^\alpha = (t, x)$ fluctuate rapidly, whereas fluctuations in the transverse plane $y^i = (y, z)$ are much slower. In such a situation we have not one but rather two relevant length scales, namely, the longitudinal and the transverse scales. Therefore we can form two dimensionless ratios:

$$\begin{aligned} g_{\parallel} &= \frac{\ell_{\text{Pl}}}{\ell_{\parallel}} \sim 1 \\ g_{\perp} &= \frac{\ell_{\text{Pl}}}{\ell_{\perp}} \ll 1. \end{aligned} \tag{1.25}$$

From now on, the first few Greek characters $\alpha, \beta, \ldots$ refer to the longitudinal space, and middle Latin letters $i, j, \ldots$ refer to the transverse plane.

Taking $\ell_{\parallel}$ to be of order ℓ_{Pl}, we are left with one dimensionless coupling:

$$\kappa = \frac{\ell_{\text{Pl}}}{\ell_{\perp}} \ll 1. \tag{1.26}$$

Performing the rescaling in the action explicitly, in four dimensions the Einstein-Hilbert action splits into three terms:

$$S[G]_{\text{EH}} = \frac{1}{8\pi G_{\text{N}}} \int \mathrm{d}^4x \sqrt{-G}\, R[G] \quad = \quad \frac{1}{\kappa^2} S_0[\hat{G}] + \frac{1}{\kappa} S_1[\hat{G}] + S_2[\hat{G}]. \tag{1.27}$$

Thus, part of the action is strongly coupled, whereas S_0 is weakly coupled. The important conclusion is that for the weakly coupled piece we can use the saddle-point approximation. As far as this part of the action is concerned, the leading contribution is given by the classical configurations. Therefore, in the limit of low-momentum transfer, high-energy amplitudes can be computed using semi-classical techniques.

Considering perturbations around a classical background,

$$g_{\mu\nu} = g^{\rm cl}_{\mu\nu}(x) + \kappa\, h_{\mu\nu}, \tag{1.28}$$

the authors of [121] found that the action reduces to

$$S_{\rm EH} = \int \sqrt{-g_{\rm cl}}\,[h^i_i K^{\alpha\beta} h_{\alpha\beta} + \frac{1}{4}\epsilon^{ik}\epsilon^{jl}\nabla_\alpha h_{ij}\nabla^\alpha h_{kl} - \frac{1}{2}(R_i + \epsilon^{\alpha\beta}\partial_\alpha h_{i\beta})^2] + \text{tot. der.} \tag{1.29}$$

where $R_i = \epsilon^{\alpha\beta}\partial_\alpha\partial_i X^a \partial_\beta X_a$ and $K^{\alpha\beta} = \nabla^\alpha\nabla^\beta - g^{\alpha\beta}_{\rm cl}\nabla^2$, and the fields X^a are to be defined below.

The field equations for the metric $g^{\rm cl}_{\mu\nu}$ are determined by the term of the action linear in the perturbation, $h_{\mu\nu}$. The lowest order term S_0 vanishes identically for solutions of the equations of motion. Verlinde and Verlinde found the following solutions to the equations of motion:

$$\begin{aligned} g^{\rm cl}_{\alpha\beta} &= \eta_{ab}\,\partial_\alpha X^a \partial_\beta X^b \\ g^{\rm cl}_{ij} &= g_{ij}(y) \\ g^{\rm cl}_{i\alpha} &= 0, \end{aligned} \tag{1.30}$$

and one also has $R_i = 0$. Notice that the action (1.29) contains no y-derivatives, and so it is like a dimensionally reduced 2-dimensional action. The X-fields then represent y-dependent displacements of the longitudinal plane into itself.

It is now convenient to define a vector field V_i^α with the following properties

$$\begin{aligned} \partial_i X^a &= V_i^\alpha \partial_\alpha X^a, \\ \partial_i g_{\alpha\beta} &= \nabla_\alpha V_{i\beta} + \nabla_\beta V_{i\alpha}. \end{aligned} \tag{1.31}$$

This vector field describes the flow of the X^a-fields in the y-direction. The action then reduces to

$$S_{\rm EH} = \int \sqrt{g_\parallel g_\perp}\left(R[g_\perp] - \epsilon_{\alpha\gamma}\epsilon_{\beta\delta}\nabla^\alpha V_i^\beta \nabla^\gamma V^{i\delta} - \frac{1}{2}(\epsilon^{\alpha\beta}\partial_\alpha V_{i\beta})^2\right). \tag{1.32}$$

This theory is topological: the first two terms can be written as a total derivative, and the last term is set to zero by the constraint $R_i = 0$.

Therefore, the action (1.32) reduces to the following boundary term:

$$S_{\rm EH} = S_{\partial M}[\bar{X}] = \int_{\partial M} \mathrm{d}x^\alpha \int \sqrt{g_\perp}\,\epsilon_{ab}\left(R[g_\perp]\bar{X}^a\partial_\alpha\bar{X}^b + \partial_i\bar{X}^a\partial_\alpha\partial^i\bar{X}^b\right) \tag{1.33}$$

where $\bar{X}$ are the boundary values of X. The boundary here corresponds to the four asymptotic null regions of the 2-dimensional Minkowski plane.

For full details, we refer to [121]. After including point particles, it turns out that the X^a's couple to the longitudinal momenta of the particles. The S-matrix computed from (1.32) with point particles gives exctly the amplitude computed by 't Hooft. In fact, quantisation of the model gives rise to the following commutator:

$$[X^a(y), X^b(y')] = i\epsilon^{ab} f(y, y') \tag{1.34}$$

where f is the Green's function.

$$(\triangle_h - \frac{1}{2}R[h])\, f(y, y') = \delta^{(2)}(y - y') \tag{1.35}$$

This is obviously 't Hooft's result (1.21).

The important conclusion of [121] is that, in the eikonal regime, quantum gravity is a topological field theory: its degrees of freedom live on the boundary, and its only physical perturbations are the global variations of the fields X^a. When coupled to point particles, the saddle-point of these variations correspond to shock waves. Indeed, after inserting the solutions (1.30), the full four-dimensional metric is :

$$\mathrm{d}s^2 \quad = \quad \eta_{ab}\, \partial_\alpha X^a \partial_\beta X^b \, \mathrm{d}x^\alpha \mathrm{d}x^\beta + g_{ij}(y)\, \mathrm{d}y^i \mathrm{d}y^j \,, \tag{1.36}$$

with

$$\begin{aligned} X^- &= x^- + p^- \theta(x^+) f(y), \\ X^+ &= x^+ - p^+ \theta(x^-) f(y), \end{aligned} \tag{1.37}$$

and this is obviously a generalisation of the Aichelburg-Sexl metric (1.12) for the case of two shock-waves[2].

In chapter 3 we will perform a systematic study of the eikonal regime, valid for spaces with a cosmological constant and of any dimension.

1.2 String Theory

Although still unsolved, Hawking's information paradox has proven to be a very useful scenario to obtain new insights that can help us construct a consistent theory of quantum gravity. Discussions about black holes have led to the discovery of several guiding principles that should be present in quantum gravity. Holography, complementarity, some sort of extendedness beyond the point particle approximation, and non-commutativity, seem to be some of the features that quantum gravity should meet. All of these are present in the eikonal regime of quantum gravity which we studied in the previous sections. In general, however, quantum gravity as a theory of point particles is quite intractable and one may need to make some additional assumption like the assumption that particles have a string-like extension. This leads us to string theory.

There are several reasons to think that making such an assumption is a good idea. Suffice it to say that string theory seems to have built in some of the above principles, in particular the principle of holography, as we will discuss in the next section.

[2]In four dimensions, there are no exact two-particle solutions known. Equation (1.36) is only valid at the linearised level.

The action of a point particle is simply given by the invariant length of its world line:

$$S = -m \int_\gamma \mathrm{d}s \sqrt{-G_{\mu\nu}(z)\, \dot{z}^\mu \dot{z}^\nu} \qquad (1.38)$$

where γ is the world line of the particle and z^μ its trajectory along this world line. It is, however, more convenient to have a quadratic action. This can be done by introducing an auxiliary field:

$$S = \frac{1}{2} \int \mathrm{d}s\, (\frac{1}{e}\, G_{\mu\nu} \dot{z}^\mu \dot{z}^\nu - e\, m^2). \qquad (1.39)$$

For a quantum mechanical particle, one integrates over all possible trajectories and also over the auxiliary field:

$$\int \mathcal{D}z \mathcal{D}e\, e^{iS[z,e]}. \qquad (1.40)$$

The saddle point approximation to the path integral selects the classical trajectory with minimal length.

For strings the situation is analogous to the point particle case. The path integral now contains the exponentiated area of the string,

$$\int \mathcal{D}X \mathcal{D}h\, e^{iS[X,h]} \qquad (1.41)$$

where $X(\tau, \sigma)$ denotes the embedding of the string into target space and h_{ij} is an auxiliary field representing the metric on the string. The action is given by

$$\begin{aligned} S &= -\frac{T}{2} \int \mathrm{d}^2\sigma\, [\sqrt{h} h^{ij} G_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu + \epsilon^{ij} B_{\mu\nu}(X) \partial_i X^\mu \partial_j X^\nu] \\ &\quad + \frac{1}{4\pi} \int \mathrm{d}^2\sigma \sqrt{h}\, \phi(X)\, R[h] \end{aligned} \qquad (1.42)$$

where we are allowing for additional background fields apart from the metric: the dilaton $\phi(X)$ and an antisymmetric tensor field $B_{\mu\nu}(X)$.

It is well known that at low energies string theory reproduces gravity. The vanishing of the β-functions of the sigma-model (1.42) imposes, at first order in α' [26]:

$$\begin{aligned} R_{\mu\nu} - \frac{1}{4} H_\mu{}^{\alpha\beta} H_{\nu\alpha\beta} + \nabla_\mu \nabla_\nu \phi &= 0 \\ \nabla^\alpha H_{\alpha\mu\nu} - 2\nabla^\alpha \phi\, H_{\alpha\mu\nu} &= 0 \\ 4(\nabla\phi)^2 - 4\Box\phi - R + \frac{1}{12} H^2 &= 0, \end{aligned} \qquad (1.43)$$

where $H_{\mu\nu\alpha}$ is the field-strentgh constructed from $B_{\mu\nu}$. The expansion parameter α' is proportional to the string length and is inversely proportional to the string tension T. The β-function equations at lowest order determine the space-time dimension, $D = 26$ for the bosonic string, and $D = 10$ for the superstring.

These equations can be integrated to the following effective action:

$$S = -\frac{1}{\kappa^2}\int \mathrm{d}^D x\,\sqrt{G}\,e^{-2\phi}[R + 4(\nabla\phi)^2 - \frac{1}{12}\,H^2]. \tag{1.44}$$

By a field redefinition of the metric one can bring the action to the Einstein frame. It is clear that higher order terms in the expansion of the β-functions (1.43) will show up as α'-corrections in the effective action (1.44). These are typically of order R^2 and higher, and they predict specific corrections to Einstein's theory.

The action (1.44) does not contain all of the massless supergravity fields. Let us for example concentrate on type IIB string theory. In this case there are additional terms one can add to the effective action. One of these is a self-dual 5-form F_5, which then gives rise to an extremal 3-brane solution of the following form:

$$\begin{aligned} \mathrm{d}s^2 &= f^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}\vec{x}_3^2) + f^{1/2}(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_5^2) \\ f &= h + \frac{R^4}{r^4} \end{aligned} \tag{1.45}$$

and $h = 1$. This solution has a constant dilaton, a covariantly constant 5-form flux along the S^5 and $H = 0$. The strentgh of the flux and the value of the dilaton are absorbed in the definition of R. However, 3-branes can also be viewed from a different point of view: they are the hyperplanes on a 10-dimensional flat space on which open (and closed) strings can end and they are called Dirichlet branes. From this point of view, one can effectively describe the physics by the effective action on the D3-brane by describing its collective modes, which are the excitations of the open strings. The effective action in the case of N D-branes placed on top of each other is the Dirac-Born-Infeld action, which generalises the world-volume action of a single D-brane and accounts for the strings being stretched between the branes.

One can also consider the above solution for $h = 0$. The space-time is then $\mathrm{AdS}_5\times S^5$. The D3-brane and the AdS metrics agree at $r/R \ll 1$, which is precisely the near-horizon limit considered in the AdS/CFT correspondence. In other words, near the horizon of the D3-brane the space looks locally like $\mathrm{AdS}_5 \times S^5$, just as the near-horizon geometry of the Schwarzschild black hole is Rindler space times a two-sphere of constant radius.

$\mathrm{AdS}_5 \times S^5$ is an exact solution of string theory, but the above extremal D3-brane metric is not. α'-corrections to the effective above action (1.44) become important as the energy increases. Here we again concentrate on the case of type IIB, which is where these corrections are best known. Keeping only the 5-form in the RR sector, the action at next order in α' is given by [61, 47, 63]:

$$S = \int \mathrm{d}^{10}x\sqrt{g}\,[e^{-2\phi}(R + 4(\partial\phi)^2 + \gamma W) - \frac{1}{2\cdot 5!}\,F_5^2], \tag{1.46}$$

where γ is a number of order α'^3. The self-duality of F_5 ensures that there are no higher order corrections in F. W is a sum of certain contractions of four Weyl tensors, $W \sim C^4$. Terms of order R^2 and R^3 are removed by a field redefinition. The Einstein frame is reached by a redefinition $g \to e^{\phi/2}g$, and the action becomes:

$$S = \int \mathrm{d}^{10}x\sqrt{g}\,[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2\cdot 5!}\,F_5^2 + \gamma e^{-\frac{3}{2}\phi}W]. \tag{1.47}$$

Branes play an essential role in the arguments leading to the AdS/CFT correspondence. Since the D3-brane is not an exact solution of the β-function equations, it would be very interesting to analyse α'-corrections to the metric (1.45). The analysis of these corrections will be presented elsewhere [34].

1.3 The AdS/CFT Correspondence

The AdS/CFT correspondence is the most concrete example of a holographic duality. It states that *string theory in an AdS space-time is equivalent to a certain conformal field theory formulated on the boundary of AdS.* So, for example, if the bulk is AdS_5, the dual CFT on the boundary is $\mathcal{N} = 4$ SYM on the boundary of AdS_5 which can be thought of as a cylinder. Of course, as it stands the formulation of this duality is still too vague. Later on we will give more details about the correspondence.

One of the surprising things about the Maldacena or AdS/CFT conjecture is that string theory contains gravity, whereas the field theory does not. This suggests that gravitational theories have redundant degrees of freedom [117] or, at least, they can be reorganised in a more economic way. This gives rise to a theory that is non-gravitational and, furthermore, is defined on a manifold with one space dimension less. In other words, gravity does not contain as many degrees of freedom as one would naively think.

The relationship between gauge theories and theories containing gravity like string theory is long standing [118]. However, a precise connection exists only since the discovery of the AdS/CFT correspondence [87, 125, 64]. There is a large literature on checks of the correspondence between supergravity in AdS and the large N limit of conformal field theories. In this thesis we will concentrate on rather generic but precise questions concerning the holographic map between both theories. Indeed, it is important to have a precise understanding of how quantities in the bulk and on the boundary are mapped into each other in order to understand how holography works.

As said, the focus will be on generic questions concerning the duality. Mostly we will not specify the details of the CFT that we are studying but assume that it exists and require minimal knowledge about it, like which sources are turned on. Then we try to reconstruct the bulk theory as far as we can with this information, until new information from the CFT is required. That we are interested in generic properties of the holographic map is due to the fact that we would like to understand holography in general, i.e. also for other backgrounds than AdS. Hopefully this will give more insight in why the duality works. In the case of the AdS/CFT correspondence, the duality between open and closed strings lies at the heart of the holographic relation [79].

Among the many phrases that can be found in the holographic dictionary, a very important notion is that of the UV/IR connection [109, 94, 13], i.e. the duality between high and low energies on both sides of the duality. More precisely, the renormalisation group scale in the gauge theory is interpreted as the compactification radius of the gravity theory. Radial evolution is then related to the renormalisation group equations [30, 119, 120].

Another, related aspect one would like to understand precisely is the geometry. How is the information about the geometry of the bulk precisely encoded in the boundary theory? More precisely, we can ask: given a certain boundary theory, how does one reconstruct the classical bulk space-time and the fields on this space-time? This an other questions will be addressed in chapter 4. Some of those results will be extended in chapter 5 to asymptotically flat and asymptotically de Sitter space-times: the information about the bulk geometry is encoded in certain specific "holographic" stress tensors.

For a review of the arguments motivating this duality, see [1]. One important issue that one has to address with any duality is its limits of validity. Indeed, string theory in AdS is usually too complicated to be dealt with in detail. One of the tractable limits is the supergravity limit where $\ell_{\rm Pl} < l_s \ll R$, which implies $1 \ll g_s N < N$. The condition $R \gg \ell_{\rm Pl}$ is needed in order for higher curvature corrections to be small. $\ell_{\rm Pl} < l_s$ is equivalent to $g_s < 1$ which is needed in order to avoid string loop corrections in the string coupling e^{ϕ}, which are not well defined in supergravity which is a non-renormalisable theory. On the SYM side this corresponds to the large N, strong 't Hooft coupling $\lambda = g_{\rm YM}^2 N$ limit of the theory. There are of course other intersting limits that one can look at but we will not consider those here.

Let us now discuss how to make the AdS/CFT correspondence more precise. In particular there is an important issue about boundary conditions at infinity that needs to be considered [64, 125]. AdS has a timelike boundary at infinity. Therefore, fields on this space can propagate to the boundary and so one has to supplement them with certain boundary conditions. There is a precise 1-1 correspondence between the boundary values of fields on AdS and operators on the CFT. We collectively denote bulk fields by Φ, and their boundary values by $\phi_{(0)}$. The string partition function is then a functional of the boundary values of the fields:

$$Z_{\rm string}[\phi_{(0)}] = \int_{\phi_{(0)}} \mathcal{D}\Phi \, \exp(-S[\Phi]). \tag{1.48}$$

According to the proposal in [125], this should be equal to the generating functional of correlation functions in the CFT,

$$Z_{\rm string}[\phi_{(0)}] = Z_{\rm CFT}[\phi_{(0)}] = \langle \exp[\int_{\partial X} {\rm d}^d x \sqrt{g}\, \phi_{(0)}(x) O(x)] \rangle_{\rm CFT} \tag{1.49}$$

where $O(x)$ is a specific composite operator in the CFT and ∂M is the boundary of the manifold M. Thus, the boundary values of the fields act as sources for computing correlation functions of operators in the CFT.

The partition function (1.48) is an intractable object to deal with, so one has to consider some limit like for example the supergravity limit. In this limit, one of the fields that will be integrated over in (1.48) is the metric. Thus, we are strictly speaking not considering AdS space, but any Einstein manifold with fixed metric at infinity. Therefore, the metric in the bulk is allowed to fluctuate as long as it preserves the boundary conditions.

Obviously, at low energies we are interested in the supergravity limit of (1.48) where the dominant contribution to the path integral is given by the saddle-point approximation. The partition function then reduces to:

$$Z_{\rm sugra}[\phi_{(0)}] = \exp(-S[\Phi_{\rm cl}(\phi_{(0)})]), \tag{1.50}$$

where $\Phi_{\rm cl}$ are now fields that satisfy the low-energy equations of motion with fixed boundary values $\Phi(r,x)|_{r=0} = \phi_{(0)}(x)$.

In general, massive scalar fields that solve the equations of motion behave differently from $\Phi(r,x) \to \phi_{(0)}(x)$ as they approach the boundary. They can either decay more rapidly or develop singularities. A more detailed analysis gives:

$$\Phi(r,x) = r^{d-\Delta}\phi_{(0)}(x) + \cdots , \tag{1.51}$$

where Δ satisfies

$$\Delta(\Delta - d) = m^2 \tag{1.52}$$

and m is the mass of the scalar field. So Δ has two possible values, $\Delta = d/2 \pm \sqrt{d^2/4 + m^2}$ which satisfy $\Delta_+ + \Delta_- = d$. This means that the expansion in general has the following asymptotic form:

$$\Phi(r,x) = r^{d-\Delta}(\phi_{(0)} + \mathcal{O}(r^2)) + r^{\Delta}(\varphi(x) + \mathcal{O}(r^2)). \tag{1.53}$$

where $\phi_{(0)}$ and φ are two independent modes. The unitarity bound on the mass implies $\Delta > (d-2)/2$. The existence of two independent solutions to the equations of motion reflects the fact that usually one needs to impose two boundary conditions on the fields: initial conditions for the positions and the momenta[3]. In AdS, usually one of these two modes will vanish if we also impose some regularity condition in the centre of AdS or some global condition like the vanishing of the Weyl tensor.

The correspondence between the gravity and the CFT computations has been tested for 2-, 3- and 4-point functions of several operators [46, 38, 1].

Klebanov and Witten have argued [81] that for $-d^2/4 < m^2 < -d^2/4 + 1$ the existence of two independent modes for fields in this mass range implies the existence of two conformal field theories dual to the same bulk metric. These are called the Δ_+ and the Δ_--theory. In the Δ_+-theory, the lowest-order mode[4] $\phi_{(0)}$ has the usual interpretation as an external source that couples to an operator $O(x)$ of conformal dimension Δ_+, whereas $\varphi(x)$ (which appears at order Δ_-) is related to the expectation value of $O(x)$. In the Δ_--theory, on the other hand, $\phi_{(0)}$ is interpreted as an expectation value whereas φ is the source. Both theories are related by a Legendre transformation. The case $\Delta_+ = \Delta_-$ is special and corresponds to the tachyon of minimal mass.

As it stands, the correspondence (1.50) is meaningless as both sides suffer from divergences. These, however, can be regularised and renormalised by adding appropriate counter-terms [70, 11, 83, 35]. It has been shown [109, 70] that the IR divergences on the gravitational side correspond to UV divergences on the gauge theory side. In chapter 4 we will develop a systematic method to regularise and renormalise the on-shell supergravity action. Although from the gravity point of view the divergences are purely classical and related to the infinite volume of AdS, it is essential to remove them in order for gravity solutions to have a sensible interpretation in terms of mass, entropy, etc. [25, 11, 83].

In chapter 4 we will study these issues in detail for scalar fields, for the metric and for the coupled gravity-matter system.

[3]However, quantisation in AdS is subtle due to the fact that there is no complete Cauchy surface [9, 50].

[4]Lowest order in r. This mode is the first mode to appear in a perturbative expansion in terms of r.