

Chapter 2

Holography in High-Energy Scattering

In this chapter we consider collisions between massless particles at very high energies. We do this perturbatively in the eikonal approximation, where collisions are almost head-on and the impact parameter is large. In this regime, gravity reduces to a topological field theory with zero bulk degrees of freedom. We discuss how to go beyond the extreme eikonal regime as well as first and second quantisation of gravitationally interacting particles.

The contents of this chapter are based mainly on [32] and [33]. The chapter is organised as follows. The first section reviews massless particle solutions of Einstein's equations for various back-ground geometries. In section 2.2 we discuss classical scattering between these particles at very high energies, and in section 2.3 we give a covariant generalisation (in transverse space) of 't Hooft's S-matrix, discussed in section 1.1.1 of the introduction. A first step towards the restoration of covariance in the longitudinal plane is taken in section 2.4 where we compute the transfer of momentum during collisions at high energies. In the next section, section 2.5, we discuss the quantum theory and find a closed algebra between momenta and a gravitational correction to Heisenberg's uncertainty which is nothing but an expression of this momentum transfer. In section 2.6 a precise link is proven between transfer of momentum and covariance. In section 2.7 we discuss second quantisation of gravitationally interacting particles and find that they satisfy an exchange algebra that is very much reminiscent of the Moyal product. We close the chapter with a discussion and some conclusions in section 2.8.

2.1 Pointlike massless particles in Einstein's theory

When energies are so high that gravity becomes the dominant force and particles start interacting gravitationally, one needs to take into account the back-reaction of particles on the back-ground geometry. That is, one cannot trust the free Einstein equations, but one has to couple them to the matter fields of the particles. Our main focus will be massless particles, as these are the relevant excitations when we discuss scattering in

the neighbourhood of a black hole.

Massless particles are included in Einstein's theory as follows. The gravitational action is given by

$$S = S_{\text{EH}} + S_{\text{matter}}, \quad (2.1)$$

where

$$S_{\text{EH}} = \frac{1}{16\pi G_{\text{N}}} \int_X d^d x \sqrt{-G} (R + 2\Lambda) \quad (2.2)$$

and S_{matter} is the matter action belonging to the massless particle. In spaces with a boundary, as is the case when the cosmological constant is negative, the action (5.2) may be supplemented with additional boundary terms to ensure a well-defined variational problem. This point will be discussed in detail in later chapters.

As is well-known, the matter action for a massless particle includes an auxiliary field e :

$$S_{\text{matter}} = -\frac{1}{2} \int ds e(s) G_{\mu\nu} \dot{z}^\mu \dot{z}^\nu. \quad (2.3)$$

Making use of the gauge invariance of the action, the equations of motion of the matter fields z^μ and of the auxiliary field give the usual geodesic equation together with the constraint that the particle is massless. Einstein's equations then take the following form:

$$R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - \Lambda G_{\mu\nu} = -8\pi G_{\text{N}} T_{\mu\nu}, \quad (2.4)$$

where the stress-energy tensor is given by:

$$T^{\mu\nu}(x) = -\frac{p}{\sqrt{-G(x)}} \int_\gamma ds \delta^{(d)}(x - z(s)) \dot{z}^\mu \dot{z}^\nu, \quad (2.5)$$

and γ is the world-line of the particle, parametrised by s , $z^\mu(s)$ the trajectory of the particle along the world-line, p the momentum of the particle along the light-cone, and d the space-time dimension. To find solutions describing massless particles, one solves (2.4) coupled to the geodesic equation and the constraint.

A useful technique to obtain solutions describing massless particles from existent vacuum solutions is Penrose's cut-and-paste method [95]. As explained in the introduction, massless particles can be seen as space-time defects of dimension 1. Penrose's method provides a space-time with a delta-function singularity with support on the null line along the particle's trajectory, given two flat pieces of Minkowski space. As shown by Dray and 't Hooft [40], this kind of gravitational solution generalises to a much larger class of asymptotically flat space-times. It can be further generalised to spaces with either positive or negative cosmological constant.

Let us start with the following rather general class of metrics:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = 2A(u, v) du dv + g(u, v) h_{ij}(x^k) dx^i dx^j, \quad (2.6)$$

in co-ordinates $x^\mu = (u, v, x^i)$. Let us assume that this metric is a vacuum solution of Einstein's equations. In this space-time, the stress tensor takes the following form:

$$T_{vv} = 4pA^2\delta(v)\delta^{(d-2)}(x) \quad (2.7)$$

for a particle travelling along the null line $v = 0$, $x^i = 0$. The cut-and-paste method suggests the following ansatz for the metric:

$$ds^2 = 2A(u, v) dv(du - f(x^k)\delta(v)dv) + g(u, v) h_{ij}(x^k) dx^i dx^j, \quad (2.8)$$

and indeed by direct computation (see Appendix A.2) one finds that Einstein's equations reduce to the following equations at $v = 0$:

$$\Delta_h f - \frac{1}{A} \partial_u \partial_v g f = \frac{gA}{\sqrt{-G}} \delta^{(d-2)}(x), \quad (2.9)$$

$$\partial_u A = \partial_u g = 0. \quad (2.10)$$

The first equation is a junction condition for gluing together both parts of the metric along the null line $v = 0$. The second equation is the requirement that the metric has a Killing vector along the null trajectory of the particle. So, two regions of space-time can be glued together only along a direction with a Killing vector.

The metric (2.8) is singular at $v = 0$. However, this singularity can be removed by a discontinuous co-ordinate transformation

$$\tilde{u} = u - f(x^k)\theta(v). \quad (2.11)$$

In these co-ordinates, the metric is finite but not continuous. Geodesics are continuous but not differentiable. With a further (continuous but not differentiable) co-ordinate transformation one can make the metric continuous.

One first remark is that the d -dimensional Einstein equations reduce to the equation of motion of a massive scalar, coupled to a source, on the space-time defect. In this case the space-time defect is a null surface of dimension 1. This result underlies the S-matrix Ansatz. In later chapters we will see that when the defect is timelike and of codimension 1, the induced equations are Einstein's equations coupled to certain stress-tensors. It would be interesting to perform a similar analysis for other types of defects, like e.g. null defects of codimension 1.

The solutions to (2.9) and (2.10) are easy to find if the background is Minkowski space. In four dimensions we find the logarithmic solution given in the introduction, equation (1.14). In other dimensions the solution generically goes like $f \sim \frac{1}{|x|^{d-4}}$. For more general backgrounds, like the Schwarzschild black hole, some solutions are given in [40].

It is not difficult to extend the above analysis to space-times including a cosmological constant [72]. Let us just give the solution for pure AdS space with a massless particle travelling from the boundary to the bulk. We write pure AdS $_d$ in the following co-ordinates (see Appendix A.1 for the transformation to Poincare co-ordinates):

$$ds^2 = \frac{4}{(1 - y^2/\ell^2)^2} \eta_{\mu\nu} dy^\mu dy^\nu, \quad (2.12)$$

where ℓ is the AdS radius and $y^2 = \eta_{\mu\nu}y^\mu y^\nu$. The stress tensor of a massless particle can straightforwardly be computed and gives $T_{uu} = -p\delta(u)\delta(\rho)$, where ρ is the radial co-ordinate $\rho = \sum_{i=1}^{d-2} y_i^2$.

This metric is not of the class considered above. However, it gives the following solution of Einstein's equations with a massless particle:

$$ds^2 = \frac{4}{(1 - y^2/\ell^2)^2} (\eta_{\mu\nu}dy^\mu dy^\nu + 8\pi G_N p_u \delta(u)(1 - \rho^2/\ell^2)f(\rho)du^2) \quad (2.13)$$

provided

$$\Delta_h f - 4 \frac{d-2}{\ell^2} f = \delta(\rho). \quad (2.14)$$

Δ_h is the Laplacian on the transverse hyperbolic space,

$$ds^2 = \frac{d\rho^2 + \rho^2 d\Omega_{d-3}^2}{(1 - \rho^2/\ell^2)^2}. \quad (2.15)$$

The solutions to (2.14) are given in chapter 3 and they of course reduce to the Minkowski solutions $f \sim \frac{1}{|x|^{d-4}}$ in the limit when the AdS radius goes to infinity, $\ell \rightarrow \infty$.

This metric can also be obtained with Penrose's method because the conformal factor has no dependence on the longitudinal co-ordinates at the locus of the shock-wave. In fact, the condition (2.14) is very similar to (2.10) and it is very likely that one can easily generalise the construction of Dray and 't Hooft to space-times with a cosmological constant that have the Horowitz-Itzhaki shock-wave as a special case. For this case one needs to introduce a dependence on the transverse length ρ in the Ansatz (2.6). Note that the effect on outgoing massless particles takes the form of a shift also in this case (see Appendix A). This can be shown either by direct computation or by using the fact that massless geodesics are invariant under Weyl rescalings of the metric. The latter fact relates the trajectories in AdS to trajectories in flat space.

It is interesting to note that shock-wave solutions are exact solutions of string theory. Indeed, in [5] it has been shown that shock-wave backgrounds are solutions to all orders in the sigma-model perturbation theory. In [72], it was shown that also the AdS shock-wave does not receive any α' -corrections from a geometrical argument used in [77, 71]. The argument uses the fact that all scalar combinations that can be formed from the contribution to the Riemann tensor due to the shock-wave vanish. Thus, corrections to the supergravity action can only come from the AdS part of the metric, but these are known to be equally zero. Thus, shock-waves are among the few known examples of exact backgrounds of string theory. Another interesting fact is that the amplitude computed by 't Hooft agrees, at large distances, with the amplitude of a free string in the shock-wave background generated by another string. The latter also agrees with the (infinite genus) amplitude of two interacting strings in a flat background. So, the shock wave can be regarded as a non-perturbative effect coming from the resummation of flat-metric string contributions [3, 4]. At small distances, however, the string amplitudes do not exhibit the singular behaviour of the point particle case. Let us however point out that to our knowledge no amplitude valid beyond the eikonal regime has been computed so far for the point particle case, and so there is not much one can conclude from the discrepancy.

2.2 Classical scattering at Planckian energies

Next we compute the effect of shock-waves on the trajectories of test particles. This is a straightforward computation if one is careful [33], although there are mathematical subtleties one has to take into account [104, 84]. We illustrate this for the case of a Minkowski background, but the computation generalises straightforwardly to other spaces. Take the metric

$$ds^2 = 2dv (du - f_v(\tilde{x}) \delta(v) dv) + dx^2 + dy^2, \quad (2.16)$$

where the shift function is $f_v(\tilde{x}) \equiv -\frac{1}{T} \int d^2\tilde{x}' P_v(\tilde{x}') f(\tilde{x} - \tilde{x}')$. This is a straightforward generalisation for the case that the total momentum is not concentrated at one point, but is a distribution over the shock-wave. This allows to describe an arbitrary amount of left-movers (see Figure 2.1) all sitting on a plane of constant v with total momentum distribution P_v . The in-going momentum distribution $P_v(\tilde{x})$ is typically equal to

$$P_v(\tilde{x}) = \sum_{i=1}^N p_v^i \delta(\tilde{x} - \tilde{x}^i), \quad (2.17)$$

if there are N particles with transverse positions x^i on the plane of the shock-wave. The right-moving particles have initial momentum p_u^0 . All particles satisfy the mass-shell condition $p_\mu^2 = 0$.

The first geodesic equation in the metric (2.16) gives

$$\ddot{v} = 0, \quad (2.18)$$

where the dot denotes the derivative with respect to the affine parameter λ along the geodesic. This equation allows us to use v as a time co-ordinate. The other equations are solved as follows:

$$\begin{aligned} u(v) &= u(0) - \frac{1}{2T} \operatorname{sgn}(v) \int d^2\tilde{x}' P_v(\tilde{x}') \left(f(\tilde{x}_0 - \tilde{x}') + v \frac{\partial x^i}{\partial v}(0) \partial_i f(\tilde{x}_0 - \tilde{x}') \right) \\ x^i(v) &= x^i(0) + p_0^i v + \frac{1}{2T} v \operatorname{sgn}(v) \int d^2\tilde{x}' P_v(\tilde{x}') \partial_i f(\tilde{x}_0 - \tilde{x}'), \end{aligned} \quad (2.19)$$

where $\tilde{x}_0 \equiv \tilde{x}(0)$. As a boundary condition, we have chosen that the initial momentum in the u -direction is zero, and in the transverse i -direction¹ it is p^i .

If we now concentrate on the $y - v$ plane, differentiating (2.19) yields

$$\frac{\partial y}{\partial v} = \frac{1}{2T} \operatorname{sgn}(v) \int d^2\tilde{x}' P_v(\tilde{x}') \partial_y f(\tilde{x}_0 - \tilde{x}'). \quad (2.20)$$

This agrees with a standard computation by Dray and 't Hooft [40] where massless geodesics are obtained from massive ones by boosting a black hole to the speed of light while sending its mass to zero.

¹The latter will be set to zero in the following.

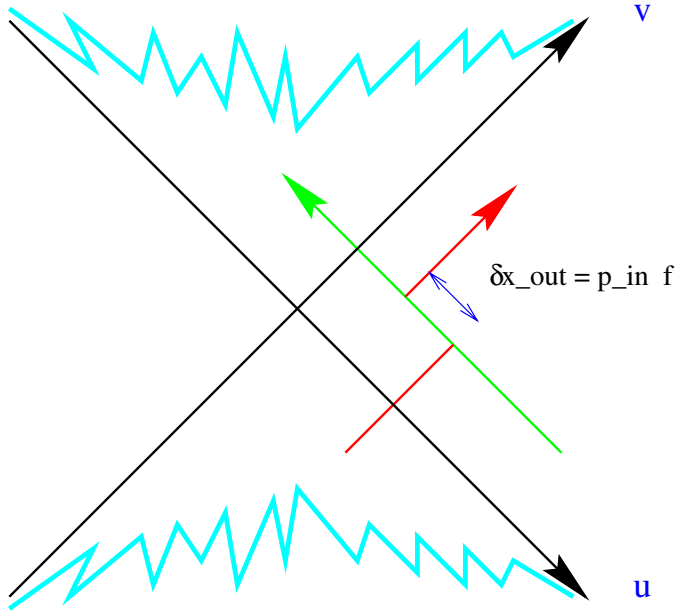


Figure 2.1: Effect of a shock-wave in the lightcone directions

As mentioned in the introduction, we will be working in the first few orders in the eikonal approximation. We introduce the expansion parameter $\varepsilon \equiv G p_{\text{in}} b$, where p_{in} is the in-going momentum and b the impact parameter, given by the transverse separation between the colliding particles. ε can be taken to be small in the eikonal regime, and it will control our perturbative expansion. Notice that, since f is logarithmic in the transverse distance, $\partial_i f \sim \frac{1}{b}$.

The first of (2.19) gives us the shift (5.2) in the longitudinal co-ordinate u as a consequence of the in-going particle plus a correction that is $\mathcal{O}(\varepsilon^2)$ and can be neglected as long as the in-going transverse momentum is small, $p_{\perp} \ll p_{\parallel}$. The second of (2.19) can be represented by a kink in the trajectory of the out-coming particle, see Figure 2.2. This is a higher-order effect.

As mentioned, these are also the trajectories in AdS with a shock-wave, (2.13), with f replaced by the corresponding shift (A.8).

So far we have discussed how the trajectories of out-coming particles are modified by the shock-waves of in-going particles. Next we will consider the momentum transfer involved.

In Figure 2.2 the trajectories in the $y-v$ plane are shown. These follow from (2.19). We learn from the figure that

$$\tan \gamma = \frac{p_y}{p_u}, \quad (2.21)$$

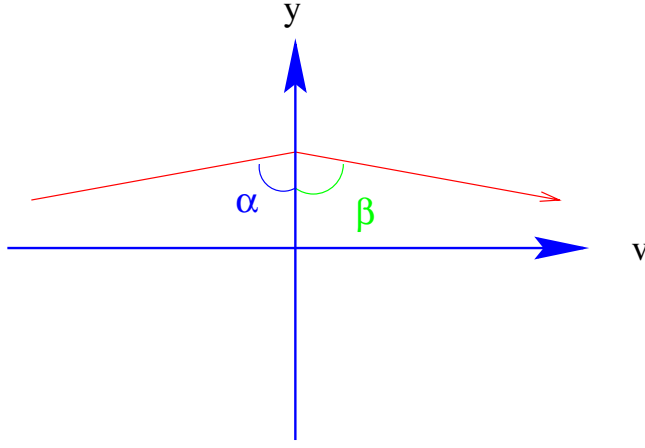


Figure 2.2: Effect of the shock-wave in the transverse direction

where the angle γ is defined by $\gamma = \pi - \alpha - \beta$, and α and β are defined as in the figure. p_y and p_u are the momentum of the out-coming particle in the y and v -directions, respectively, *after* it passes the shock wave. These quantities are different from the momenta before the interaction, which we denote by p_μ^0 . We do not explicitly write the superscripts in or out, as it should be clear from the context whether the momentum refers to the in-going or out-coming particle².

We now take the initial transverse momentum to be zero, $p_y^0 = 0$. This means that $\alpha = \pi/2$ and hence, from (2.20),

$$\cot \alpha + \cot \beta = \tan \gamma = \frac{1}{T} p_v \partial_y f(y_0). \quad (2.22)$$

One can easily check that the exchange of momentum in the v -direction, to first order in ε , is equal to zero and hence $p_u \simeq p_u^0$. Therefore, we have

$$p_y = \frac{1}{T} p_u p_v \partial_y f. \quad (2.23)$$

Since $p_y \sim \frac{\partial y}{\partial v}$, this can also be directly deduced from (2.20).

If the initial transverse momentum is nonzero, differentiating (2.19) once yields, at $v > 0$,

$$\frac{\partial u}{\partial v} = -\frac{1}{2T} \frac{\partial x^i}{\partial v} p_v \partial_i f(\tilde{x}_0), \quad (2.24)$$

so for the out-coming particle we have

$$p_v^{\text{out}} = -\frac{1}{T} p_i^{0,\text{out}} p_v^{\text{in}} \partial^i f(\tilde{x}_0). \quad (2.25)$$

²In the remainder of this section we assume there is only one particle coming in.

The same relation is obtained from the mass-shell condition $p_\mu p^\mu = 0$.

From (2.23) and (2.25) we find that, roughly speaking, $\delta p_\perp \sim p_\parallel \varepsilon$ and $\delta p_\parallel \sim p_\perp \varepsilon$, and so if $p_\perp \ll p_\parallel$, the transfer of momentum in the transverse plane is much larger than in the longitudinal plane.

Furthermore, as the transfer of momentum in both the longitudinal and the transverse plane are $\mathcal{O}(\varepsilon)$, they are negligible for large transverse separations (compared to the Planck length). That is the regime where the eikonal approximation is valid.

2.3 The S-matrix

The classical trajectories found in section 2.2 are enough to obtain the scattering amplitude of two particles in the eikonal approximation. In this approximation, the net effect of the presence of a shock-wave on another particle is a shift of the corresponding wave-function. Naively one would think that since the whole effect is only a shift, it can be gauged away with a suitable choice of co-ordinates. However, as argued in section 1.1.2, although locally on both sides of the shock-wave there is no effect, there is an important global effect which is the shift. This shift cannot be removed by a co-ordinate transformation, despite the suggestive form of the metric (1.36). This can be more easily understood in analogy with the electromagnetic case [113, 76]. When a charged particle is boosted towards the speed of light, the electromagnetic field A_μ of the particle is pure gauge outside the light-cone of the particle, i.e. $A_\mu = \frac{1}{2} \partial_\mu \Lambda$ and so has no net physical effect there. However, the gauge field is discontinuous along the world line of the particle, and so the transformations needed to gauge it away are different on the future and past light-cones, $A_\mu^\pm = \pm \frac{1}{2} \partial_\mu \Lambda$ with $\Lambda = Q/2\pi \log|x|$, Q being the charge of the particle. Therefore, the total effect is physical.

As explained in the introduction, the scattering amplitude computed from the shift (1.16) is the Veneziano amplitude. The effective action that one finds after a Fourier transformation of the amplitude is

$$S = \int d^2 \tilde{x} (-T \partial_i u \partial^i v + P_u u - P_v v). \quad (2.26)$$

This is nothing but a rewriting of (1.20) for a flat background. As remarked in the introduction, this is the action of a non-linear sigma model with a coupling to an external source P_μ . The coupling constant is $T = \frac{1}{8\pi G_N}$. The equations of motion following from this action directly lead to the geodesic equation in the eikonal approximation:

$$\begin{aligned} \partial_i^2 u(\tilde{x}) &= \frac{1}{T} P_v(\tilde{x}) \\ \partial_i^2 v(\tilde{x}) &= -\frac{1}{T} P_u(\tilde{x}), \end{aligned} \quad (2.27)$$

which are solved by

$$\begin{aligned} u(\tilde{x}) &= u_0 - \frac{1}{T} \int d^2 \tilde{x}' P_v(\tilde{x}') f(\tilde{x} - \tilde{x}') \\ v(\tilde{x}) &= v_0 + \frac{1}{T} \int d^2 \tilde{x}' P_u(\tilde{x}') f(\tilde{x} - \tilde{x}'). \end{aligned} \quad (2.28)$$

We first write these equations according to a 2+2-splitting of space-time. This is easy to do in the longitudinal plane. We find:

$$X^a(\sigma) = x^a - \frac{1}{T} \epsilon^{ab} \int d^2\sigma' P_b(\tilde{\sigma}) f(\tilde{\sigma} - \tilde{\sigma}'). \quad (2.29)$$

The quantisation of this model has been discussed in (1.21). We get:

$$[X^a(\sigma), X^b(\sigma')] = -\frac{1}{T} \epsilon^{ab} f(\tilde{\sigma} - \tilde{\sigma}'). \quad (2.30)$$

Notice that the minus sign difference in (2.28) is crucial to obtain the epsilon tensor. Indeed, had we guessed a relation of the type $\partial_i^2 x^a \sim p^a$, then the right-hand side of (2.30) would not have been antisymmetric and the model would have been inconsistent at the quantum level³. Indeed, due to the complete symmetry between u and v , one's naive guess would have been a geodesic equation where both terms in (2.27) have the same sign. However, the minus sign is directly linked to causality: one of the particles is in-going, whereas the other is out-going. We will comment some more on this in the conclusion.

The presence of an epsilon-tensor is also proven in [121] from the manipulations of the Einstein-Hilbert action coupled to massless particles in the eikonal limit, as reviewed in the introduction.

Recall that the equation of motion (2.27) is only valid in Minkowski space. Indeed, when the manifold is curved the shift function f gets a mass term as shown in (2.9), and this is the equation one has to take as a starting point for more general backgrounds. It can be expressed in terms of the induced metric h_{ij} if one considers that the second term on the left-hand side of (2.9) is a relic of the two-dimensional Ricci-tensor. From the computation outlined in Appendix A.2, we find that the Ricci tensor of the vacuum metric (2.6) equals

$$R_{ij}[G] = R_{ij}[h] - \frac{\partial_u \partial_v g}{A} h_{ij}, \quad (2.31)$$

where $R_{ij}[G]$ is the transverse part of Ricci tensor obtained from the full four-dimensional Riemann tensor, see (A.13), and $R_{ij}[h]$ is the Ricci tensor corresponding to the two-dimensional metric h_{ij} . Since $R_{ij}[G]$ satisfies the vacuum Einstein equations, the constraint reduces to

$$R[h] = \frac{2}{A} \partial_u \partial_v g \quad (2.32)$$

for a two-dimensional metric h . This obviously gives the metric on the sphere if $g = r^2$ and $A = 1$. We can write equation (2.9) as

$$\left(\Delta_h - \frac{1}{2} R[h] \right) f = \frac{1}{\sqrt{h}} \delta^{(2)}(\tilde{x} - \tilde{x}_0). \quad (2.33)$$

³Of course, the sign can be reabsorbed in the definition of momentum, but this leads to non-standard commutation relations and is therefore not very useful for the discussion of covariant generalisations.

It is now obvious how to include this extra term in (2.27):

$$(\Delta_h - \frac{1}{2} R[h]) X^a = \frac{1}{2T} \epsilon^{ab} P_b. \quad (2.34)$$

This equation is reminiscent of the focusing theorem. It is solved exactly as before,

$$X^a(\sigma) = x^a + \frac{1}{2T} \int d^2 \tilde{\sigma} \sqrt{\tilde{h}} \epsilon^{ab} P_b(\sigma') f(\tilde{\sigma} - \tilde{\sigma}'), \quad (2.35)$$

where f is now the solution of the generalised Green equation (2.33).

It is now straightforward to find a ‘‘covariant’’ generalisation of the action (2.26) in the eikonal limit:

$$S = -\frac{T}{2} \int d^2 \sigma \sqrt{h} [h^{ij} \partial_i X^a \partial_j X_a + \frac{1}{2} R[h] X^a X_a + \frac{1}{T} \epsilon^{ab} X_a P_b]. \quad (2.36)$$

We put the word ‘‘covariant’’ between quotation marks because the fields X^a are still two-dimensional as we are still in the eikonal limit. Thus, covariance here is only with respect to the transverse co-ordinates.

An alternative way to derive this equation is by performing a Fourier transformation of the amplitude (1.19) with a generalised shift that satisfies (2.33). In the case that the metric h is the metric on the unit sphere, we get the amplitude (1.19) computed for the Schwarzschild background.

Let us consider the symmetries of (2.36) for a moment. First of all there is the Lorentz symmetry which we just referred to. It is interesting to note that this symmetry is induced by time translations in Rindler time, as shown in (1.9). Thus, one can say that time translations in the bulk induce Lorentz boosts on the boundary. This is very reminiscent of the relation between radial translations in the bulk and conformal transformations on the boundary for the case of AdS, although the groups are obviously different. It will be interesting to investigate the symmetries of the boundary action in the case of AdS.

The original 't Hooft action (2.26) was invariant under Weyl rescalings of the boundary metric. In the general case we find that the term in (2.36) proportional to the curvature explicitly breaks this symmetry and we are left with a global symmetry only.

2.4 The eikonal limit and beyond

We have mentioned that shock-wave solutions are exact solutions of Einstein’s equations, even if one includes any higher-curvature corrections, like for example the ones that appear in string theory. These come from the conformal invariance of the sigma-model. Furthermore one can compute the exact effect of the shock-wave on outgoing particles and the transfer of momentum. Therefore one can ask the question: what happens when one increases the energy up to the Planck scale and perhaps beyond? In other words, how does one go beyond the eikonal approximation? This is the question we are going to analyse in detail in this section.

't Hooft has suggested [115, 113] that a covariant generalisation of the equations of motion (2.34) should automatically account for the transfer of momentum⁴. However, as we will explain later, it is extremely difficult to find a consistent generalisation of this formula in four dimensions.

Instead, we will choose another approach here. In (2.23) and (2.25) we found the exact momentum transfer. These formulae indeed hold without any approximations. So we will write these formulae in a manifestly covariant form, and will then discuss quantisation. The covariant expression will automatically account for the transfer of momentum. In this section we will study this formula in detail, and in later sections we show that it is consistent with quantisation. Let us first give the expression:

$$P_{\text{out}}^\mu(\tilde{\sigma}) = (g^{\mu\nu} + A^{\mu\nu})P_\nu^{0,\text{out}}(\tilde{\sigma}), \quad (2.37)$$

where

$$A^{\mu\nu}(\tilde{\sigma}) = -\frac{1}{T} \epsilon^{\mu\nu\lambda\rho} \epsilon^{ij} \partial_i X_\lambda(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_{\rho,\text{in}}(\tilde{\sigma}') \partial_j f(\tilde{\sigma} - \tilde{\sigma}'). \quad (2.38)$$

For the in-operators, one interchanges the labels in-out in the above expression. The quantities P_{out}^0 and P_{out} are the momenta of the out-coming particle before and after the interaction, respectively.

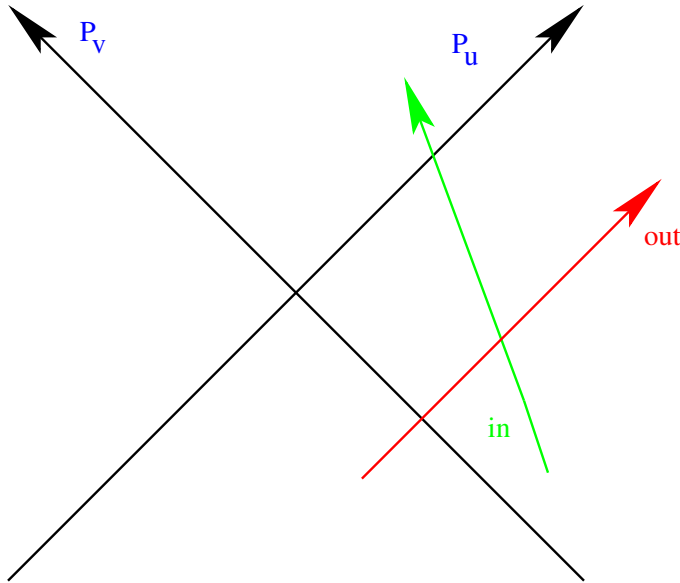


Figure 2.3: Collision at non-zero angle

⁴By covariant we really mean covariant with respect to 4-dimensional diffeomorphisms.

Let us now check that this covariant expression reproduces the momentum transfer computed before. To that end, we first have to set up some notation. We use the notation of reference [121], explained in section 1.1.2 of the introduction. Four-dimensional fields X^μ split into a longitudinal and a transverse component, $X^\mu = (X^a, Y^m)$. The internal co-ordinates x^α and $y^i \equiv \sigma^i$ are (in the usual gauge) the zero modes of X^a and Y^m , respectively, roughly: $X^a = x^\alpha + \dots$ and $Y^m = \sigma^i + \dots$. In view of this, and since we will be making a distinction between (X, Y) and (x, y) , the indices a and α can be identified, and also m and i (but notice that $X^\alpha \neq x^\alpha, Y^i \neq \sigma^i$). We analyse the case when the background is Minkowski. Both for the out- and the in-particles we have $P_{\parallel} = P_\alpha = (P_u, P_v)$, $P_\perp = P_i$. We will consider the change of momentum for the out-going particles produced by the in-going particles, but the expressions for the in-going particles are trivially obtained by exchanging the labels “in” and “out”. The kinematics is illustrated in Figure 2.3.

There is still a point in using this (2+2)-splitting of space-time even if the transverse momentum is not zero, because the longitudinal and transverse momenta behave differently in the first few orders in the eikonal approximation. We get:

$$\begin{aligned}
P_i^{\text{out}}(\tilde{\sigma}) &= P_i^{0,\text{out}}(\tilde{\sigma}) + \frac{1}{T} P_v^{0,\text{out}}(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_u^{\text{in}}(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}') \\
&- \frac{1}{T} P_u^{0,\text{out}}(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_v^{\text{in}}(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}') + \mathcal{O}(\varepsilon^2) \\
&= P_i^{0,\text{out}}(\tilde{\sigma}) + \frac{1}{T} \varepsilon^{ab} P_a^{0,\text{out}} \int d^2\tilde{\sigma}' P_b^{\text{in}} \partial_i f(\tilde{\sigma} - \tilde{\sigma}'). \tag{2.39}
\end{aligned}$$

Notice that if the operator on the left-hand side of (2.39) carries an out-label, then the operator on the right-hand side of (2.41) which is evaluated at $\tilde{\sigma}$ corresponds to the same out-particle, whereas the operators which are integrated over give the contributions from the in-particles. The same is true if one reverses the labels.

Note that even if the initial transverse momentum is zero, like in head-on collisions, it will be non-vanishing after the interaction. The two particles will spin around each other for a short time. This agrees with equations (2.23) and (2.25), which were obtained from kinematical considerations. If the momentum of the out-going particle satisfies $p_{\parallel}^{\text{out}} \gg p_{\perp}^{\text{out}}$, using the equation for the shift

$$X^a(\tilde{\sigma}) = x^a + \frac{1}{T} \int d^2\sigma' \varepsilon^{ab} P_b(\sigma') f(\sigma - \sigma') \tag{2.40}$$

we find from (2.39)

$$P_i(\tilde{\sigma}) = P_i^0(\tilde{\sigma}) + P_a^0 \partial_i X^a, \tag{2.41}$$

to first order in ε . This expression was found in [113] from the consideration that the transverse momentum is not an independent variable, together with the requirement that the transverse momentum generates transverse translations. Here we see that it straightforwardly follows from the transfer of momentum during the collision.

The transverse momentum (2.39) can also be written as

$$P_i(\tilde{\sigma}) = P_i^0(\tilde{\sigma}) + \frac{\varepsilon}{T} P_a^0(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_0^a(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}'), \tag{2.42}$$

where $\epsilon = 1$ if P_i is an operator corresponding to the in-going particles and $\epsilon = -1$ for the out-operators. This is the usual sign convention, where all in-going momenta are defined to be positive, and out-coming momenta to be negative [113]. Indeed, if initially the in-going particles only have momentum P_v , and the out-coming ones only momentum P_u , (2.42) gives

$$\begin{aligned} P_i^{\text{in}}(\tilde{\sigma}) &= P_i^{0,\text{in}}(\tilde{\sigma}) + \frac{1}{T} P_{v,0}^{\text{in}}(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_u^{\text{out}}(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}') \\ P_i^{\text{out}}(\tilde{\sigma}) &= P_i^{0,\text{out}}(\tilde{\sigma}) - \frac{1}{T} P_{u,0}^{\text{out}}(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_v^{\text{in}}(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}'). \end{aligned} \quad (2.43)$$

The next nontrivial check concerns the longitudinal momentum transfer. Using (2.37), we find

$$\begin{aligned} P_u(\tilde{\sigma})^{\text{out}} &= P_u^{0,\text{out}}(\tilde{\sigma}) - \frac{1}{T} P_0^i \int d^2\tilde{\sigma}' P_u \partial_i f + \frac{1}{T} P_u^0 \int d^2\tilde{\sigma}' P^i \partial_i f, \\ P_v^{\text{out}}(\tilde{\sigma}) &= P_v^{0,\text{out}}(\tilde{\sigma}) + \frac{1}{T} P_0^i \int d^2\tilde{\sigma}' P_v \partial_i f - \frac{1}{T} P_v^0 \int d^2\tilde{\sigma}' P^i \partial_i f. \end{aligned} \quad (2.44)$$

In covariant (2+2)-notation,

$$\begin{aligned} P_{\text{out}}^a(\tilde{\sigma}) &= P_{0,\text{out}}^a(\tilde{\sigma}) + \frac{1}{T} \epsilon^{ab} P_b^{0,\text{out}}(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_{\text{in}}^i(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}') \\ &\quad - \frac{1}{T} \epsilon^{ab} P_{0,\text{out}}^i(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_b^{\text{in}}(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}') \\ &= P_{0,\text{out}}^a(\tilde{\sigma}) - P_{0,\text{out}}^i(\tilde{\sigma}) \partial_i X^a(\tilde{\sigma}) \\ &\quad + \frac{1}{T} \epsilon^{ab} P_{b,\text{out}}^0(\tilde{\sigma}) \int d^2\tilde{\sigma}' P_{\text{in}}^i(\tilde{\sigma}') \partial_i f(\tilde{\sigma} - \tilde{\sigma}'). \end{aligned} \quad (2.45)$$

Again, it perfectly agrees with (2.25) in the corresponding limit, $P_{\text{in}}^i = 0$. Notice that the transfer of longitudinal momentum is zero if the initial transverse momentum is zero. So, although for vanishing initial transverse momentum there is still a transverse momentum transfer, in the longitudinal plane this transfer is zero to first order in ϵ .

Notice that in the situation that is usually considered, $P_{\text{in}}^i = 0$, equations (2.41) and (2.45) can be rewritten as

$$\begin{aligned} \delta P_i &= +W_i^a P_a^0 \\ \delta P^a &= -W_i^a P^{i,0}. \end{aligned} \quad (2.46)$$

where we defined $W_i^a = \partial_i X^a$. Since W_i^a is proportional to the vector field V_i^α in formula (1.31) by $W_i^a = V_i^\alpha \partial_\alpha X^a$, we see that the latter is responsible for the transfer of momentum. The analogy with fluid dynamics suggested in [121] becomes more transparent from this computation: this vector field accounts for the “flow” or “vorticity” of momentum during collisions at high energies.

2.5 Quantisation

A full quantum theory for this non-linear four-dimensional model is extremely difficult to write down away from the eikonal limit. To quantise the theory we have to give a

complete set of observables and the way they act on states in Hilbert space. Now in this model the space-time co-ordinates are not independent, but are related by shift equations. Upon expanding the fields into eigenmodes of this equation of motion with the corresponding creation and annihilation operators, we will find that co-ordinates do not commute. This result has been known for a long time (see [113] and references therein). Quantisation in the eikonal limit is quite straightforward and gives rise to non-commuting co-ordinates (2.30). However, beyond the eikonal approximation we encounter non-linearities which are difficult to deal with. We anticipate that we will not be able to give an exhaustive list of commutation rules among all operators beyond the eikonal limit, for basically the same reason it was not found in earlier works [113, 33]. Instead, we will give a complete set of commutators between the momenta, P . These commutators of course satisfy the Jacobi identity. It is however not clear how to derive a well-defined commutator between the X 's. One would think it can be obtained from a generalisation of (2.35) or (2.30), but this is not straightforward as these equations become highly non-linear at low impact parameter. Part of the problem also stems from the fact that the commutator is non-local and so does not transform properly under non-linear co-ordinate transformations of the world-sheet co-ordinates. In fact, it is not even clear whether X^μ is a good starting point to define a quantum theory that incorporates non-linear effects, as it does not transform as a vector in target space. 't Hooft has stressed [113] that a consistent quantisation scheme can perhaps be found if one introduces variables that are better behaved. In the next section we will see the derivative of X , ∂X , is a better physical observable. In this section we will concentrate on the operator P , which is also well behaved as it is a natural object of the tangent space.

As said, here we will assume the canonical commutator between position and momentum operators, and find commutation relations for the momenta from the momentum transfer equation (2.37). We will see that one does find a set of commutation rules that is consistent, where the momentum operator has the usual interpretation as the generator of translations. We will find that the commutators appearing in [115], postulated from the condition that momentum operators generate translations, automatically follow from our equations of motion. We also find new commutators which close the algebra of momenta.

As for the commutators between the X 's, these should follow from the Jacobi identity. Indeed, in three dimensions we have solved the Jacobi identity and recovered the results which were found in [116] and [33] directly from the shock-wave equations of motion. However, we have not been able to integrate the equation in four dimensions.

We first consider the action of the operators \hat{P}^μ and \hat{X}^μ on state vectors $|P_0\rangle$ and $|X\rangle$. We obviously have

$$\begin{aligned}\hat{P}^\mu|P_0\rangle &= P_0^\mu|P_0\rangle; \\ \hat{X}^\mu|X\rangle &= X^\mu|X\rangle.\end{aligned}\tag{2.47}$$

Furthermore, the operators \hat{P} and \hat{X} satisfy the usual commutation relation

$$[\hat{X}^\mu(\tilde{\sigma}), \hat{P}^\nu(\tilde{\sigma}')] = ig^{\mu\nu} \delta(\tilde{\sigma} - \tilde{\sigma}').\tag{2.48}$$

Indeed, at the level of the path integral and in the eikonal limit these operators were

related by a Fourier transformation [32]. From now on we drop the carets on operator-valued quantities.

The quantum theory will however be an interacting theory, and has to take (2.37) into account. Therefore, just as in the eikonal limit (2.29) was promoted to an operator identity, leading to (2.30), our assumption will be that (2.37) is also a relation between a free operator P_0 and the interacting field P . We get the following modified commutator:

$$[X^\mu(\tilde{\sigma}), P^\nu(\tilde{\sigma}')] = i(g^{\mu\nu} + A^{\mu\nu}) \delta(\tilde{\sigma} - \tilde{\sigma}'). \quad (2.49)$$

This simply means that, due to the back-reaction, the number of independent measurements one can do simultaneously is reduced according to:

$$\Delta x \Delta p \geq \frac{\hbar}{2} + \mathcal{O}\left(\frac{\ell_{\text{Pl}}^2 p_{\text{in}}}{b}\right). \quad (2.50)$$

The modification of the canonical commutation relation (2.48) in the presence of gravitational interactions has also been predicted (although in different contexts) by several authors [86, 78, 124].

The generalised commutator (2.49) has a simple interpretation if we go back to the underlying shock-wave picture. Before the interaction takes place, the different momenta are independent variables. However, *after* the interaction, they are coupled through the momentum transfer equation (2.37). Then the longitudinal momenta generate sideways displacements as well. So it is natural to identify the canonical momentum P_{can}^μ with the momentum before the interaction, which we denote by P_0^μ , and P^μ with the momentum after the collision. The latter describes the momentum transfer, and can be seen to be a measure for the recoil of the particles. This holds both for the in-going and the out-coming particles. Although P^μ is not a canonical operator, when writing (2.49) out in components we will see that it generates translations in the sense of field theory. The situation here is similar to cases with background electromagnetic fields, take for example a particle in an electromagnetic field. In that case, the kinetical momentum, which is the operator that (by Ehrenfest's theorem) satisfies the classical equation of motion, is not the canonical momentum operator.

Let us now take a closer look at the commutation relation (2.49),

$$[X^\mu(\tilde{\sigma}), P^\nu(\tilde{\sigma}')] = iG^{\mu\nu} \delta(\tilde{\sigma} - \tilde{\sigma}'), \quad (2.51)$$

where the “generalised metric” is defined as

$$G^{\mu\nu} \equiv g^{\mu\nu} + A^{\mu\nu}. \quad (2.52)$$

Writing (2.51) out in components, we find

$$\begin{aligned} [u(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= i\partial_i u \delta(\tilde{\sigma} - \tilde{\sigma}'); \\ [v(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= i\partial_i v \delta(\tilde{\sigma} - \tilde{\sigma}'); \\ [Y^m(\tilde{\sigma}), p_u(\tilde{\sigma}')] &= i\partial_u Y^m \delta(\tilde{\sigma} - \tilde{\sigma}'); \\ [Y^m(\tilde{\sigma}), p_v(\tilde{\sigma}')] &= i\partial_v Y^m \delta(\tilde{\sigma} - \tilde{\sigma}'). \end{aligned} \quad (2.53)$$

In the 2+2 splitting, this can be reexpressed as

$$\begin{aligned} [X^a(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= i\partial_i X^a(\tilde{\sigma}) \delta(\tilde{\sigma} - \tilde{\sigma}') \\ [Y^m(\tilde{\sigma}), p_\alpha(\tilde{\sigma}')] &= i\partial_\alpha Y^m(\tilde{\sigma}) \delta(\tilde{\sigma} - \tilde{\sigma}'). \end{aligned} \quad (2.54)$$

In the gauge where longitudinal indices α are along X^a , and transverse indices i are along Y^m , defining $\delta X^a = X^a - x^a$ and $\delta Y^m = Y^m - \sigma^m$, we have:

$$\partial_\alpha \delta Y_i + \partial_i \delta X_\alpha = 0. \quad (2.55)$$

Note that the operators P_μ are not usual translation operators. They rather generate translations of the fields X^μ along the internal directions.

In quantum mechanics, co-ordinates are independent of each other, and so the right-hand side of (2.51) reduces to the canonical commutator $ig^{\mu\nu} \delta(\tilde{\sigma} - \tilde{\sigma}')$. But in our case we have a two-dimensional field theory where the longitudinal and the transverse co-ordinates become mutually dependent fields. This renders (2.51) non-vanishing even if the indices μ and ν are different (notice that, for $\mu \neq \nu$, (2.51) is nonzero if one of the indices is transverse, say i , and the other one is a longitudinal index α). So p generates translations just as in field theory, as one directly sees from (2.53).

One can also get an algebra for the commutator of the p 's among themselves. One finds (the operators referring all to the in- or all to the out-states)

$$\begin{aligned} [p_\alpha(\tilde{\sigma}), p_i(\tilde{\sigma}')] &= ip_\alpha(\tilde{\sigma}') \partial_i \delta(\tilde{\sigma} - \tilde{\sigma}'); \\ [p_i(\tilde{\sigma}), p_j(\tilde{\sigma}')] &= ip_i(\tilde{\sigma}') \partial_j \delta(\tilde{\sigma} - \tilde{\sigma}') + ip_j(\tilde{\sigma}) \partial_i \delta(\tilde{\sigma} - \tilde{\sigma}'), \end{aligned} \quad (2.56)$$

Now we can also obtain an algebra that relates the in- and the out-operators. Using (2.56), we get:

$$\begin{aligned} [p_v^{\text{in}}(\tilde{\sigma}), p_i^{\text{out}}(\tilde{\sigma}')] &= -iT \partial_i u(\tilde{\sigma}') f^{-1}(\tilde{\sigma} - \tilde{\sigma}'); \\ [p_u^{\text{out}}(\tilde{\sigma}), p_i^{\text{in}}(\tilde{\sigma}')] &= iT \partial_i v(\tilde{\sigma}') f^{-1}(\tilde{\sigma} - \tilde{\sigma}'); \\ [p_i^{\text{in}}(\tilde{\sigma}), p_j^{\text{out}}(\tilde{\sigma}')] &= -iT \partial_i v(\tilde{\sigma}) \partial_j u(\tilde{\sigma}') f^{-1}(\tilde{\sigma} - \tilde{\sigma}') \\ &\quad + \frac{i}{T} p_v^{\text{in}}(\tilde{\sigma}) p_u^{\text{out}}(\tilde{\sigma}') \partial_j f(\tilde{\sigma} - \tilde{\sigma}'). \end{aligned} \quad (2.57)$$

In reference [113] it was not possible to find correct expressions for the commutators between in- and out-operators. The expected expression for the last of (2.57) did not satisfy the Jacobi identity when combined with (2.56). One can check that the above expression does satisfy the Jacobi identity.

The algebra (2.57) is very non-local and, furthermore, non-linear. It, however, can be significantly simplified by defining the total momentum

$$P_\mu = \int d^2\tilde{\sigma} p_\mu(\tilde{\sigma}). \quad (2.58)$$

This leads to the following local expressions:

$$\begin{aligned} [p_i^{\text{in}}(\tilde{\sigma}), P_j^{\text{out}}] &= i\partial_j p_i^{\text{in}}(\tilde{\sigma}); \\ [p_i^{\text{out}}(\tilde{\sigma}), P_j^{\text{in}}] &= i\partial_j p_i^{\text{out}}(\tilde{\sigma}); \\ [p_\alpha^{\text{in}}(\tilde{\sigma}), P_i^{\text{out}}] &= i\partial_i p_\alpha^{\text{in}}(\tilde{\sigma}); \\ [p_\alpha^{\text{out}}(\tilde{\sigma}), P_i^{\text{in}}] &= i\partial_i p_\alpha^{\text{out}}(\tilde{\sigma}), \end{aligned} \quad (2.59)$$

so the total transverse momentum generates translations.

One can check that the algebra between the transverse in-operators or the out-operators is similar to (2.59). However, we do not expect the theory to have two different generators of transverse translations. So we expect

$$\delta P_{\text{in}}^i = \delta P_{\text{out}}^i. \quad (2.60)$$

Integrating (2.39) we indeed see that this is the case. The same holds for the lightcone directions, as one sees from equation (2.44). Therefore, for the integrated momentum operators we get the constraint

$$\delta P_{\text{in}}^\mu = \delta P_{\text{out}}^\mu. \quad (2.61)$$

Recalling that these operators give the momentum transfer, this is nothing else than the expression of the conservation of momentum. As a constraint on Hilbert space, in our case it is also equivalent to the usual asymptotic completeness [48] of the in- and out-Hilbert spaces.

Equation (2.61) implies that momentum is a globally conserved quantity. But locally it is not conserved, as one can see from the individual local expressions. Only after integrating over $\bar{\sigma}$ the total momentum is conserved. This is also the usual expectation in field theory.

Recalling that we started off regarding the Minkowski plane as the near-horizon region of a Schwarzschild black hole, we have shown that one can go beyond the eikonal approximation and compute the momentum transfer, thereby respecting momentum conservation which is a minimal requirement for the unitarity of the S-matrix. The assumption that the S-matrix is unitary was the starting point of 't Hooft's considerations, as explained in the introduction. We now see that this assumption leads to a consistent algebra of momenta. In fact, it would be interesting to take the algebra (2.56)-(2.57) as the starting point of some field theory, the momentum being related to the stress-energy tensor in the usual way, and to study the Hilbert-space structure of this theory.

Since the results presented in this section are valid to the first non-trivial order in the eikonal approximation, it seems that the framework developed in [121] would be most appropriate to do an additional check of our results, and would perhaps provide some more conceptual insight in the near-eikonal regime of quantum gravity.

Some of the results in this section had already been found in [115] from general considerations. Here we learn that they straightforwardly follow when recoil effects are taken into account. Furthermore, we also get the additional equations (2.57) and (2.59), which close the algebra. In the next section we perform another check of (2.37).

2.6 Quantum gravity in 2+1 dimensions

When looking for a formulation of the S-matrix beyond the eikonal approximation, in four dimensions one encounters several problems [113, 33] that originate in the non-linearity of the equations. Indeed, as the dimension increases the equations become more and more non-linear [32]. However, when one reduces to 2+1 dimensions things simplify considerably as the algebra becomes linear.

In this section we compactify one of the space-time (and world-sheet) directions on a small circle of radius R and assume the three remaining fields X^μ to be independent of this internal dimension. We also assume that the momentum along this direction is zero and hence we only take the zero modes into account. For a complete theory one should of course also consider the excited modes.

There are several ways to set up the theory. In references [116, 33] it was chosen to find the commutator for the X fields from a covariant generalisation of the dimensionally-reduced system. Reference [116] wrote the covariant formula only after deriving the commutators, whereas in [33] the equation of motion for X was first covariantly generalised, and from there the commutator was found. Both approaches gave the same result. In reference [33] the commutator between the X 's and P 's was then found from the Jacobi identity, and it was checked that it agrees with the commutator one finds if one directly dimensionally reduces (2.49). It was concluded that the covariant generalisation of the algebra is directly related to the transfer of momentum. This also served as a check of the four-dimensional algebra, which, as stressed in [33], is not free of problems.

Here we choose an alternative route, which, as we will see, is equivalent to that of [33] and provides a nice check of our formulae in the previous section. We take the expression for the commutator between X and P , equation (2.49), as our starting point for the dimensional reduction. In four dimensions this expression comes from the transfer of momentum, (2.37). We then use the Jacobi identity to find the commutator between the X 's.

As said, both methods give the same results. The advantage of the latter method is that it only assumes transfer of momentum and not knowledge of the commutator between the X 's. Furthermore, in principle this method can be generalised to higher dimensions, where we can compute the transfer of momentum as in the previous sections, but we do not have a fully consistent equation of motion for X for the reasons explained in [33]. Solving the Jacobi identity should give the equation of motion for X . Nevertheless, we have not been able to find a solution to the Jacobi identity in four dimensions, although we do not see any reason why it should not have a solution.

We parametrise the compactified dimension by $\sigma_2 = y$, $0 \leq y \leq R_3$. We define $\sigma = \sigma_1$, $\partial = \frac{\partial}{\partial \sigma}$ and $\epsilon_{\mu\nu\lambda} = \epsilon_{\mu\nu\lambda y}$. Notice that the effective 2+1-dimensional Newton's constant is obtained from the 3+1-dimensional one by

$$G_3 = \frac{G_4}{R}. \quad (2.62)$$

Therefore we will find an effective coupling $T = RT_4$.

Since the momentum $p^\mu(\sigma_1, \sigma_2)$ is a momentum density, we have to integrate over the internal direction to obtain the observable momentum from the three-dimensional point of view: $P^\mu(\sigma_1) = \int d\sigma_2 p^\mu(\sigma_1, \sigma_2)$.

In 2+1 dimensions, (2.49) becomes

$$[X^\mu(\sigma), P^\nu(\sigma')] = i \left(g^{\mu\nu} - \frac{1}{T} \epsilon^{\mu\nu\lambda} \int d\sigma'' P_\lambda(\sigma'') \partial f(\sigma - \sigma'') \right) \delta(\sigma - \sigma') \quad (2.63)$$

where the shift function is now given by $f(\sigma - \sigma') = \frac{1}{2}|\sigma - \sigma'|$.

We can obtain the commutator between two X 's from the Jacobi identity. As remarked in [33] and stressed in previous sections, it is better to consider its derivative,

∂X^μ , rather than X itself, because the former satisfies a local algebra. So we work out the following relation:

$$[[\partial X^\mu(\sigma), P^\nu(\sigma')], \partial X^\lambda(\sigma'')] + \text{cyclic} = 0. \quad (2.64)$$

We get the following solution:

$$[\partial X^\mu(\sigma), \partial X^\nu(\sigma')] = -\frac{i}{T} \epsilon^{\mu\nu\lambda} g_{\lambda\rho} \partial X^\rho(\sigma) \delta(\tilde{\sigma} - \tilde{\sigma}'). \quad (2.65)$$

This is the $\text{SO}(2,1)$ algebra obtained in [116, 33].

Following [116], the presence of the delta-function in (2.65) suggests to define the following integrated variables:

$$x_A^\mu = \int_A d\sigma \partial x^\mu = x^\mu(A_1) - x^\mu(A_0), \quad (2.66)$$

where A is an interval $A = [A_0, A_1]$ along the line σ .

These variables have the nice property:

$$[x_A^\mu, x_A^\nu] = -\frac{i}{T} \epsilon^{\mu\nu\lambda} g_{\lambda\rho} x_A^\rho. \quad (2.67)$$

As argued by 't Hooft, this gives rise to a time variable that is quantised in units of t_{Pl}/R .

Another useful quantity is the total momentum flowing through A ,

$$p_A^\mu \equiv \int_A d\sigma P^\mu(\sigma). \quad (2.68)$$

The commutator then becomes

$$[x_A^\mu, p_A^\nu] = iG^{\mu\nu}, \quad (2.69)$$

with the ‘‘generalised metric’’

$$G^{\mu\nu} = g^{\mu\nu} - \frac{1}{T} \epsilon^{\mu\nu\lambda} g_{\lambda\rho} p_A^\rho. \quad (2.70)$$

The same results can be derived [33] from a covariant generalisation of the three-dimensional equation of motion (2.35):

$$\partial^2 x^\mu = \frac{1}{T} \epsilon^{\mu\nu\lambda} g_{\lambda\rho} \partial x^\rho p_\nu. \quad (2.71)$$

One can also work out the commutation relations in a way analogous to equations (2.53)-(2.54), finding that P again has an interpretation as the generator of translations:

$$\begin{aligned} [u_A, p_x^A] &= i(\partial u)_A \\ [v_A, p_x^A] &= i(\partial v)_A, \end{aligned} \quad (2.72)$$

etc., in a way analogous to the 3+1-dimensional case.

It is particularly beautiful that the link between a covariant algebra, where all coordinates are treated on the same footing, and the inclusion of the transverse gravitational force, can be made so precise in 2+1 dimensions: including the transverse gravitational force leads to an algebra that is invariant under the full three-dimensional Lorentz group, and viceversa: writing the algebra in a manifestly $\text{SO}(2,1)$ invariant form automatically accounts for transverse effects.

2.7 Second quantisation of gravitationally interacting particles

Gravitational interactions at high energies lead to a non-commutative space-time. One can wonder what consequences this has for fields that live on this space-time. In reference [80], it was found that taking into account the back-reaction of particles on a black-hole horizon leads to quantised fields that satisfy a so-called exchange algebra. This exchange algebra exhibits great similarity with the Moyal product defined in non-commutative gauge theories.

The computation of [80] uses 't Hooft's results to model a forming black hole with a horizon that fluctuates in time. The formation of the future horizon depends on the time of arrival of in-coming particles, and thus it matters whether we first add in-going particles and then measure the positions of out-going particles, or viceversa.

In this section we show that this effect is not at all an exclusive feature of time-dependent black holes (although black-holes are the natural scenario where these effects become important). Gravitationally interacting fields in Minkowski space already obey such an exchange algebra if they interact gravitationally. All the considerations in this section are independent of the dimension, except for the details of the eikonal approximation. This section is based on [39].

Consider two massless particles in Minkowski space. Particle 1 is “hard” and carries a shock-wave with it, whereas particle 2 is “soft” and so its back-reaction can be neglected. Particle 1 is a left-mover with momentum k^- along x^- , and particle 2 is a right-mover with momentum k^+ along x^+ . When particle 2 crosses the trajectory of particle 1 at $x^+ = 0$, it will get shifted:

$$\delta x^- = k^- f, \quad (2.73)$$

and the impact parameter is kept fixed.

Next we consider quantised fields in this Minkowski background. For the moment we restrict ourselves to fields with no transverse momentum. These fields fall apart into a + and a - component:

$$\phi(x^+, x^-) = \phi_+(x^+) + \phi_-(x^-). \quad (2.74)$$

Therefore, the Hilbert space decomposes into a left- and a right-moving part.

To have an S-matrix description, we must have some notion of asymptotic states. Because of (2.74), the Hilbert space of the in-states will fall apart into:

$$|\text{in}\rangle_- |\text{in}\rangle_+, \quad (2.75)$$

and likewise for the out-states. The S-matrix will relate both sets of states. In a momentum representation, if there are for example N in-going particles with momentum along the x^- -direction, we have a state

$$|k_1^-, \dots, k_N^-\rangle_{\text{in}, -}. \quad (2.76)$$

We now consider creation and annihilation operators of particles at I_- and I_+ . A creation operator $a_+^\dagger(k^+)$ that naturally acts on an in-state is defined by

$$a_+^\dagger(k^+) |0\rangle_{\text{in}, +} = |k^+\rangle_{\text{in}, +}, \quad (2.77)$$

and likewise for the x^- -direction (see Figure 2.4). We require these operators to satisfy the usual commutation rules

$$[a_\alpha(k), a_\beta^\dagger(k')] = \delta(k - k') \delta_{\alpha\beta}, \quad (2.78)$$

where the Greek indices stand for $+$ or $-$ ⁵. For the out-states we have a similar definition, and the corresponding operators will be called $b_\alpha(k)$.

We now consider the commutation rules between in- and out-operators. In the absence of any interactions, the S-matrix is simply unity and so the Hilbert spaces $|\text{in}\rangle_+$ and $|\text{out}\rangle_+$ are identified. a_α and b_β then satisfy

$$\begin{aligned} [a_+(k), b_+^\dagger(k')] &= \delta(k - k') \\ [a_-(k), b_-^\dagger(k')] &= \delta(k - k'). \end{aligned} \quad (2.79)$$

The $+$ and the $-$ -operators mutually commute in this case.

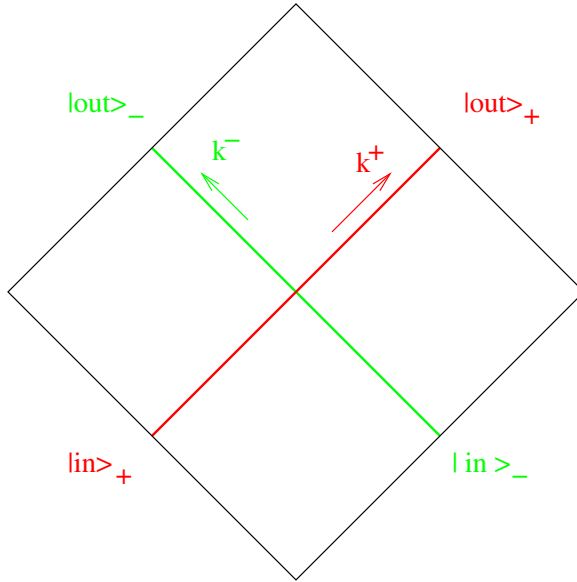


Figure 2.4: Asymptotic states in a two-particle collision

We now include the gravitational interaction (2.73). We assume that the operators

⁵This definition is slightly different from, but completely equivalent to, the usual Fock space. In usual Fock space, the states are characterised by the occupation numbers $|\{n_k\}\rangle$. This is a more economic arrangement of the state (2.76), but it is not useful for our purposes.

a_- and a_+ , and b_- and b_+ , will still commute⁶. Furthermore, since in the shock-wave approximation there are no self-interactions, (2.79) still holds. When transforming a state $|\text{in}\rangle_+$ into a state $|\text{out}\rangle_+$, the S-matrix element is still trivial.

The interaction (2.73) gives something non-trivial when one considers the commutators $[a_+, b_-]$ and $[a_-, b_+]$. In these cases, one has to take into account the shift (2.73).

The proposal is that in-going operators act also on the Hilbert space of out-going particles. When we add an in-going particle with momentum k^- , we are also shifting the trajectories of out-going particles with momentum along x^+ by the amount (2.73). So we define the operator $a_-(k)$ to act on out-states as follows:

$$a_-^\dagger(k)|0\rangle_{\text{out},-}|k_1^+, \dots, k_N^+\rangle_{\text{out},+} = \exp\left[-i\sum_{i=1}^N k_i^+ k^- f(\tilde{x} - \tilde{x}^i)\right] \times \\ \times |k^-\rangle_{\text{out},-}|k_1^+, \dots, k_N^+\rangle_{\text{out},+}. \quad (2.80)$$

So this operator translates out-coming particles by the corresponding shift. One can check that the states created by a_-^\dagger form a natural set of states for the out-going Hilbert space. Indeed, solving the Klein-Gordon equation in a shock-wave geometry one finds that the complete set of wave-functions are not simply plane waves, but rather plane waves translated over the corresponding shift. Equation (2.80) also defines the S-matrix.

Notice that the shift f depends on the transverse position of each particle, but this has no meaning in a momentum representation, where we have taken $\tilde{k} \approx 0$, since in principle such a particle cannot be localised. However, one can neglect this effect as long as the transverse distances are large, so that quantum fluctuations are small. As soon as the transverse distance becomes small, one also has to take transverse momentum transfer into account, and the eikonal approximation (2.80) is no longer valid.

We are now in a position to compute the difference between the products $a_-^\dagger b_+^\dagger$, $b_+^\dagger a_-^\dagger$:

$$b_+^\dagger(k) a_-^\dagger(p) |k^1, \dots, k^N\rangle_{\text{out},+}|p^1, \dots, p^M\rangle_{\text{out},-} = \\ = \exp\left[-i\sum_{i=1}^N k_i^+ p^- f(\tilde{x} - \tilde{x}^i)\right] \times \\ \times |k, k^1, \dots, k^N\rangle_{\text{out},+}|p, p^1, \dots, p^M\rangle_{\text{out},-}; \\ a_-^\dagger(p) b_+^\dagger(k) |k^1, \dots, k^N\rangle_{\text{out},+}|p^1, \dots, p^M\rangle_{\text{out},-} = \\ = \exp\left[-i\sum_{i=1}^N k_i^+ p^- f(\tilde{x} - \tilde{x}^i) - ik^+ p^- f\right] \times \\ \times |k, k^1, \dots, k^N\rangle_{\text{out},+}|p, p^1, \dots, p^M\rangle_{\text{out},-}. \quad (2.81)$$

Since the states considered here are arbitrary, we conclude that

$$b_+^\dagger(k^+) a_-^\dagger(k^-) = \exp[-ik^+ k^- f] a_-^\dagger(k^-) b_+^\dagger(k^+). \quad (2.82)$$

⁶This is actually different from the philosophy advocated by 't Hooft, who considered non-vanishing commutators for operators at spacelike separated distances, although still preserving causality. Notice, however, that even if two operators act at the same point of the light-cone $x^+ - x^-$, they still can be separated by a spacelike distance since there is still a large transverse separation $\tilde{x} - \tilde{y}$. The extended nature of the shock-wave makes it impossible to avoid non-locality.

Of course, the commutation rules for the annihilation operators can be obtained by replacing $k \rightarrow -k$.

Notice that the exchange factor (2.82) generates shifts both in the $+$ and in the $-$ direction, depending on the state it acts on. Therefore we require that b_- and a_+ obey the same algebra:

$$a_+^\dagger(k^+)b_-^\dagger(k^-) = \exp[-ik^+k^-f] b_-^\dagger(k^-)a_+^\dagger(k^+). \quad (2.83)$$

One can now define scalar fields $\phi(x^\pm)_{\text{in},\pm}$, $\phi(x^\pm)_{\text{out},\pm}$ in terms of these operators:

$$\begin{aligned} \phi_{\text{in},+}(x^+) &= \int dk_+ a_-(k_+) e^{ik_+x^+} \\ \phi_{\text{out},-}(x^-) &= \int dk_- b_+(k_-) e^{ik_-x^-}, \end{aligned} \quad (2.84)$$

and analogously for the other two fields. Notice that, since we integrate over positive and negative frequencies, these fields are automatically real and contain both creation and annihilation modes. As remarked before, one can also check that they satisfy the Klein-Gordon equation in the shock-wave geometry:

$$[\partial_+\partial_- - p^-f(\tilde{x} - \tilde{x}')\delta(x^+)\partial_-^2] \phi(x) = 0, \quad (2.85)$$

and we have neglected transverse derivatives which give factors quadratic in \tilde{k} and $\partial_{\tilde{x}}f$, which are assumed to be small. In this approximation, the solution to this equation is:

$$\phi(x) = \int dk_- d\tilde{k} F(k_-, \tilde{k}) \exp[ip^-k_- \theta(x^+)f(\tilde{x}) + ik_-x^- + ik_+x^+ + i\tilde{k} \cdot \tilde{x}], \quad (2.86)$$

where $k_+ = -\tilde{k}^2/k_-$ and under the assumption that the main contribution to the integral comes from the region of small \tilde{k} . The function F is arbitrary, and has to be fixed by imposing some boundary conditions on the field and its derivative.

To simplify notation, we write (2.84) as

$$\begin{aligned} \phi_{\text{in}}(x^+) &= \int dk_+ a(k_+) e^{ik_+x^+} \\ \phi_{\text{out}}(x^-) &= \int dk_- b(k_-) e^{ik_-x^-}. \end{aligned} \quad (2.87)$$

Now these fields satisfy the following exchange algebra:

$$\phi_{\text{out}}(x^-)\phi_{\text{in}}(x^+) = \exp[if\partial_+\partial_-] \phi_{\text{in}}(x^+)\phi_{\text{out}}(x^-), \quad (2.88)$$

which looks like the $M \rightarrow \infty$ limit of the algebra obtained in [80].

Ultimately we would like to consider not only zero modes but rather fields with transverse momentum. Indeed, the transverse distance has not properly been taken care of in (2.88). f depends on the transverse separation of ϕ_{in} and ϕ_{out} , so it is clear that the fields should depend on the transverse co-ordinates too. It is straightforward

to include transverse momentum as long as we are in the eikonal regime. Consider fields like in (2.86),

$$\phi_{\text{in}}(x) = \int dk_+ d\tilde{k} a(k_+, \tilde{k}) e^{ik_+ x^+ + ik_- x^- + i\tilde{k}\tilde{x}}, \quad (2.89)$$

where $k_- = -\tilde{k}^2/k_+$. This expression is valid as long as we consider large transverse separations between the fields. One gets:

$$\phi_{\text{out}}(y)\phi_{\text{in}}(x) = \exp\left[if^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right]\phi_{\text{in}}(x)\phi_{\text{out}}(y), \quad (2.90)$$

where $f^{\mu\nu} = \epsilon^{\mu\nu}f(\tilde{x} - \tilde{y})$, the indices running over the light-cone directions only. The epsilon-tensor is due to the minus sign coming from the antisymmetry under interchange of the in- and out-labels, and x^+ and x^- in (2.90).

Notice that, by assumption, the main contribution to the integral over transverse momenta comes from the region of small \tilde{k} . This would seem to imply that the effect of the shift of the in-field in the x^+ -direction is negligible, since its momentum $k^- = -\tilde{k}^2/k_+$ is small. However, this is not necessarily true as the factor appearing in the exponential is proportional to $-\tilde{k}^2 p^+ f(\tilde{x} - \tilde{y})/k_+$, so in 4-dimensional Minkowski space, where $f \sim \log|\tilde{x} - \tilde{y}|$, this need not be small for large transverse separations and large momenta p_+ . In other words, the eikonal approximation only requires the derivative of f to be small, but the shift itself can be large in Planck units.

The above expression is suspiciously similar to the Moyal product that one gets in non-commutative field theory. The obvious guess is that this is related to our original commutator

$$[x^\mu, y^\nu] = if^{\mu\nu}. \quad (2.91)$$

Notice, however, that the situation here is slightly different from that in non-commutative field theory in that our back-ground is commutative now. Non-commuting particle coordinates have been replaced by non-commuting fields.

2.8 Discussion and conclusions

The eikonal regime turns out to be a very interesting corner of the moduli space of quantum gravity. Things simplify so enormously in this regime that the theory becomes topological. Still it has non-trivial dynamics. Global variations of the fields correspond to massless particles in the bulk, similarly to the way massive particles in 2+1 dimensions correspond to topological defects. If the holographic principle is to be true, one should not be surprised by this conclusion but should rather wonder whether the same is true away from the eikonal regime.

It is also found that the theory is a non-commutative theory whose natural length scale is the Planck length. Furthermore, Heisenberg's relation is modified by a term proportional to Newton's constant. This has been proposed by other authors [86, 78, 124], but in the context of collisions between particles at high energies it appears to be a simple consequence of the entanglement between the particles after interactions.

Attempts to construct the S-matrix for a two-particle collision at arbitrary angles and arbitrary momentum transfer have failed so far. Note that for this it is not at all necessary to have a two-particle solution of Einstein's equations as long as the rest mass of the particles is small, as one can always go to a frame where the momentum of one of the particles is small. We have performed a somewhat indirect analysis. The momentum transfer between the particles was computed, and from this it is easy to obtain the commutation rules between momenta. At every stage conservation of momentum was explicit. However, we were not able to find an algebra between the co-ordinates, although in principle this can be found by integrating the Jacobi identity. In 2+1 dimensions, this is easy to do and we get an SO(2,1) algebra between position operators, in agreement with earlier works [116, 33]. In the derivation given here it is clear that these complicated quantum mechanical effects are again rooted in the entanglement between the particles produced by the momentum transfer.

The precise analysis of the momentum transfer also gives interesting insights in the decoupling of the longitudinal and transverse degrees of freedom. Transverse momentum transfer is of the order $\delta p_{\perp} \sim p_{\parallel} \varepsilon$, whereas longitudinal momentum transfer is of the order $\delta p_{\parallel} \sim p_{\perp} \varepsilon$. Thus, as long as $p_{\perp} \ll p_{\parallel}$, transverse physics is frozen and the transverse modes can be treated classically, whereas the longitudinal modes are still rapidly fluctuating. In fact, we have explicitly seen that when this condition is no longer valid, the transverse modes start fluctuating and become quantum mechanical operators as well.

A case of particular interest is AdS. Based on the AdS-shock-wave solution of by Horowitz and Itzhaki, we found that also in AdS interactions between massless particles are given in terms of shifts. In particular, for scalar fields the effect is a phase shift. In the next chapter we will find that, from the CFT point of view, the dual operator has a different expectation value inside the light-cone from its value outside.

Another interesting result concerns second quantisation of these gravitationally interacting particles. They satisfy an exchange algebra which is very similar to the Moyal product defined in non-commutative gauge theories, with the difference that in our case the θ -parameter is a function of the transverse co-ordinates. The non-commutativity of the algebra is rooted in the non-commutativity of the first-quantised space-time. However, despite the similarity the situation is different from that in non-commutative gauge theories as our algebra is not derived from an action on a non-commutative space. In our case the co-ordinates commute. The non-commutativity arises when we include gravitational interactions. This is true both for the first and the second quantised system: in our case, it is always the matter fields of particles that are non-commuting. These are either co-ordinates of particles, $X^{\mu}(\tilde{\sigma})$, or scalar fields, $\phi(x)$. The underlying space-time ($\tilde{\sigma}$ or x , respectively) is always commutative.

There are several interesting open questions which are left for future study.

One interesting problem is how to fully take into account the transverse effects in the non-commutative scalar field theory without having to restrict ourselves to the eikonal approximation. This could be done most easily in the 2+1-dimensional context where we have the full commutator between the x 's, which satisfy the SO(2,1) algebra, and it may be easier to obtain the exact solutions to the Klein-Gordon equation.

Another crucial question in the context of holography is the interpretation of the commutators (2.91) and (2.90) in the context of the AdS/CFT duality. The compu-

tation of the trajectories in the AdS-shock-wave metric in Appendix A.1 reveals that once again the shift is proportional to the momentum, and so upon quantisation one expects co-ordinates to be non-commuting. However, one now has to take into account the additional problems with quantisation that arise in AdS. It is likely that the techniques developed in chapter 4 can help us understand the meaning of the commutator (2.91) in terms of sources or operators related to the bulk fields $z^\mu(s)$. Another, more straightforward approach, will be to directly study the algebra (2.90) from the point of view of the dual operators on the boundary. A previous step in this direction is taken in section 3.7 of the next chapter.

As remarked by 't Hooft [116], the epsilon-tensor in (2.67) is directly related to the position of the observer with respect to a black hole horizon. In turn, the appearance of such an epsilon tensor can be traced back to the minus sign difference in (2.27), which gives rise to an epsilon tensor in the longitudinal space in formula (2.30). At the level of the S-matrix, this sign difference comes from the fact that particles with momentum p_u are in-going, whereas those with momentum p_v are out-going, as one easily sees from (1.20). Thus, this epsilon tensor is indeed connected with the distinction between in-going and out-going and thus with causality. It is at first somewhat surprising that the same epsilon tensor appears in (1.33) and (1.34), but also here it has to do with the orientation with respect to the asymptotic boundary of the space-time, and thus again it is a global property closely related with causality.

Let us end with a somewhat speculative remark. Beyond the eikonal approximation, although there may still be some hidden redundancy in our formulae, we have seen that there are more than two fields X^a whose variations contain physical information. In four dimensions, there are four such fields, X^a and Y^m . One is therefore led to speculate that the path-integral approach in [121] at the next order in the eikonal approximation will still be topological, the physical fields now being the boundary values of X^a and Y^m . Of course, at some point one expects to encounter the usual non-renormalisable infinities in quantum gravity, and at that point one may need to invoke string theory.