

Chapter 4

Holographic Reconstruction of Space-time in the AdS/CFT Correspondence

The contents of this chapter are based on [35]. For a review and explicit examples see also [101], and for a short account of the AdS/CFT correspondence see section 1.3 of the introduction and also reference [1].

We develop a systematic method for renormalising the AdS/CFT prescription for computing correlation functions. This involves regularising the bulk on-shell supergravity action in a covariant way, computing all divergences, adding counter-terms to cancel them and then removing the regulator. We explicitly work out the case of pure gravity up to six dimensions and of gravity coupled to scalars, but the techniques can be easily applied for other matter fields. The method can also be viewed as providing a holographic reconstruction of the bulk space-time metric and of bulk fields on this space-time, out of conformal field theory data. Knowing which sources are turned on is sufficient in order to obtain an asymptotic expansion of the bulk metric and of bulk fields near the boundary to high enough order so that all infrared divergences of the on-shell action are obtained. To continue the holographic reconstruction of the bulk fields one needs new CFT data: the expectation value of the dual operator. In particular, in order to obtain the bulk metric one needs to know the expectation value of stress-energy tensor of the boundary theory. We provide completely explicit formulae for the holographic stress-energy tensors up to six dimensions. We show that both the gravitational and matter conformal anomalies of the boundary theory are correctly reproduced. We also obtain the conformal transformation properties of the boundary stress-energy tensors.

4.1 Introduction and summary of the results

Holography states that a $(d+1)$ -dimensional gravitational theory¹ (referred to as the bulk theory) should have a description in terms of a d -dimensional field theory (referred to as the boundary theory) with one degree of freedom per Planck area [112, 105]. The arguments leading to the holographic principle use rather generic properties of gravitational physics, indicating that holography should be a feature of any quantum theory of gravity. Nevertheless it has been proved a difficult task to find examples where holography is realised, let alone to develop a precise dictionary between bulk and boundary physics. The AdS/CFT correspondence [87] provides such a realisation [125, 109] with a rather precise computational framework [64, 125]. It is, therefore, desirable to sharpen the existing dictionary between bulk/boundary physics as much as possible. In particular, one of the issues one would like to understand is how space-time is built holographically out of field theory data.

The prescription of [64, 125] gives a concrete proposal for a holographic computation of physical observables. In particular, the partition function of string theory compactified on AdS spaces with prescribed boundary conditions for the bulk fields is equal to the generating functional of conformal field theory correlation functions, the boundary value of fields being now interpreted as sources for operators of the dual conformal field theory (CFT). String theory on anti-de Sitter (AdS) spaces is still incompletely understood. At low energies, however, the theory becomes a gauged supergravity with an AdS ground state coupled to Kaluza-Klein (KK) modes. On the field theory side, this corresponds to the large N and strong 't Hooft coupling regime of the CFT. So in the AdS/CFT context the question is how one can reconstruct the bulk space-time out of CFT data. One can also pose the converse question: given a bulk space-time, what properties of the dual CFT can one read off?

The prescription of [64, 125] equates the on-shell value of the supergravity action with the generating functional of connected graphs of composite operators, see (1.49)-(1.50). Both sides of this correspondence, however, suffer from infinities —infrared divergences on the supergravity side and ultraviolet divergences on the CFT side. Thus, the prescription of [64, 125] should more properly be viewed as an equality between bare quantities. One needs to renormalise the theory to obtain a correspondence between finite quantities. It is one of the aims of this chapter to present a systematic way of performing such a renormalisation.

The CFT data² that we will use are: which operators are turned on, and what is their vacuum expectation value. Since the boundary metric (or, more properly, the boundary conformal structure) couples to the boundary stress-energy tensor, the reconstruction of the bulk metric to leading order involves a detailed knowledge of the way the energy-momentum tensor is encoded holographically. There is by now an extended literature on the study of the stress-energy tensor in the context of the AdS/CFT correspondence starting from [11, 91]. We will build on these and other related works [43, 88, 83].

¹In this and the next chapter, we use the convention the boundary is d -dimensional whereas the bulk is $(d+1)$ -dimensional

²We assume that the CFT we are discussing has an AdS dual. Our results only depend on the space-time dimension and apply to all cases where the AdS/CFT duality is applicable, so we shall not specify any particular CFT model.

Our starting point will be the calculation of the infrared divergences of the on-shell gravitational action [70]. Minimally subtracting the divergences by adding counter-terms [70] leads straightforwardly to the results in [11, 43, 83]. After the subtractions have been made one can remove the (infrared) regulator and obtain a completely explicit formula for the expectation value of the dual stress-energy tensor in terms of the gravitational solution.

We will mostly concentrate on the gravitational sector, i.e. on the reconstruction of the bulk metric, but we will also discuss the coupling to scalars. Our approach will be to build perturbatively an Einstein manifold of constant negative curvature (which we will sometimes refer to as an asymptotically AdS space) as well as a solution to the scalar field equations on this manifold out of CFT data. The CFT data we start from is what sources are turned on. We will include a source for the dual stress-energy tensor as well as sources for scalar composite operators. This means that in the bulk we need to solve the gravitational equations coupled to scalars given a conformal structure at infinity and appropriate Dirichlet boundary conditions for the scalars. It is well-known that if one considers the standard Euclidean AdS (i.e., with isometry $SO(1, d+1)$), the scalar field equation with Dirichlet boundary conditions has a unique solution. In the Lorentzian case, because of the existence of normalisable modes, the solution ceases to be unique. Likewise, the Dirichlet boundary condition problem for (Euclidean) gravity has a unique smooth solution (up to diffeomorphisms) in the case the bulk manifold is topologically a ball and the boundary conformal structure is sufficiently close to the standard one [58]. However, given a boundary topology there may be more than one Einstein manifold with this boundary. For example, if the boundary has the topology of $S^1 \times S^{d-1}$, there are two possible bulk manifolds [69, 125]: one which is obtained from standard AdS by global identifications and is topologically $S^1 \times R^d$, and another, the Schwarzschild-AdS black hole, which is topologically $R^2 \times S^{d-1}$.

We will make no assumption on the global structure of the space nor its signature. The CFT should provide additional data in order to retrieve this information. Indeed, we will see that only the information about the sources leaves undetermined the part of the solution which is sensitive to global issues and/or the signature of space-time. To determine that part one needs new CFT data. To leading order these are the expectation values of the CFT operators.

In particular, in the case of pure gravity, we find that generically a boundary conformal structure is not sufficient in order to obtain the bulk metric. One needs more CFT data. To leading order one needs to specify the expectation value of the boundary stress-energy tensor. Since the gravitational field equation is a second order differential equation, one may expect that these data are sufficient in order to specify the full solution. However, higher point functions of the stress-energy tensor may be necessary if higher derivatives corrections such as R^2 -terms are included in the action. We emphasise that we make no assumption about the regularity of the solution. Under additional assumptions the metric may be determined by fewer data. For example, as we mentioned above, under certain assumptions on the topology and the boundary conformal structure one obtains a unique smooth solution [58]. Another example is the case when one restricts oneself to conformally flat bulk metrics. Then a conformally flat boundary metric does yield a unique bulk metric, up to diffeomorphisms and global identifications [102].

Turning things around, given a specific solution, we present formulae for the expectation values of the dual CFT operators. In particular, in the case the operator is the stress-energy tensor, our formulae have a “dual” meaning [11]: both as the expectation value of the stress-energy tensor of the dual CFT and as the quasi-local stress-energy tensor of Brown and York [25]. We provide very explicit formulae for the stress-energy tensor associated with any solution of Einstein’s equations with negative constant curvature.

Let us summarise these results for space-time dimension up to six³. The first step is to rewrite the solution in the Graham-Fefferman co-ordinate system [44]

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{r^2} (dr^2 + g_{ij}(x, r) dx^i dx^j), \quad (4.1)$$

where

$$g(x, r) = g_{(0)} + r^2 g_{(2)} + \dots + r^d g_{(d)} + h_{(d)} r^d \log r^2 + \mathcal{O}(r^{d+1}). \quad (4.2)$$

The logarithmic term appears only in even dimensions. l is a parameter of dimension of length related to the cosmological constant as $\Lambda = -\frac{d(d-1)}{2l^2}$. Any asymptotically AdS metric can be brought in the form (4.1) near the boundary ([58], see also [59, 57]). Once this co-ordinate system has been reached, the expectation value of the boundary stress-energy tensor reads

$$\langle T_{ij} \rangle = \frac{dl^{d-1}}{16\pi G_N} g_{(d)ij} + X_{ij}[g_{(n)}], \quad (4.3)$$

where $X_{ij}[g_{(n)}]$ is a function of $g_{(n)}$ with $n < d$. Its exact form depends on the space-time dimension and it reflects the conformal anomalies of the boundary conformal field theory. In odd (boundary) dimensions, where there are no gravitational conformal anomalies, X_{ij} is equal to zero. The expression for $X_{ij}[g_{(n)}]$ for $d = 2, 4, 6$ can be read off from the formulae that will be given in (4.33), (4.38) and (4.39), respectively. The universal part of (4.3) (i.e. with X_{ij} omitted) was obtained previously in [91]. Actually, to obtain the dual stress-energy tensor it is sufficient to only know $g_{(0)}$ and $g_{(d)}$ as $g_{(n)}$ with $n < d$ are uniquely determined from $g_{(0)}$, as we will see. The coefficient $h_{(d)}$ of the logarithmic term in the case of even d is also directly related to the conformal anomaly: it is proportional to the metric variation of the conformal anomaly.

It was pointed out in [11] that this prescription for calculating the boundary stress-energy tensor provides also a novel way, free of divergences⁴, of computing the gravitational quasi-local stress-energy tensor of Brown and York [25]. Conformal anomalies reflect infrared divergences in the gravitational sector [70]. Because of these divergences one cannot maintain the full group of isometries even asymptotically. In particular, the isometries of AdS that rescale the radial co-ordinate (these correspond to dilations in the CFT) are broken by infrared divergences. Because of this fact, bulk solutions that

³In this chapter the dimension we refer to is the dimension of the boundary. So, $d = 6$ corresponds to asymptotically AdS₇.

⁴We emphasise, however, that one has to subtract the logarithmic divergences in even dimensions in order for the stress-energy tensor to be finite.

are related by diffeomorphisms that yield a conformal transformation in the boundary do not necessarily have the same mass. Assigning zero mass to the space-time with boundary R^d , one obtains that, due to the conformal anomaly, the solution with boundary $R \times S^{d-1}$ has non-zero mass. This parallels exactly the discussion in field theory. In that case, starting from the CFT on R^d with vanishing expectation value of the stress-energy tensor, one obtains the Casimir energy of the CFT on $R \times S^{d-1}$ by a conformal transformation [27]. The agreement between the gravitational ground-state energy and the Casimir energy of the CFT is a direct consequence of the fact that the conformal anomaly computed by weakly coupled gauge theory and by supergravity agree [70]. It should be noted that, as emphasised in [11], agreement between gravity/field theory for the ground state energy is achieved only after all ambiguities are fixed in the same manner on both sides.

A conformal transformation in the boundary theory is realised in the bulk as a special diffeomorphism that preserves the form of the co-ordinate system (4.1) [75]. Using these diffeomorphisms one can easily study how the (quantum, i.e., with the effects of the conformal anomaly taken into account) stress-energy tensor transforms under conformal transformations. Our results, when restricted to the cases studied in the literature [27], are in agreement with them. We note that the present determination is considerably easier than the one in [27].

Let us briefly discuss in more detail how conformal invariance is broken. As is well-known [87], the bulk metric does not quite induce a metric on the boundary, but only a conformal class of metrics. Since the metric has a double pole on the boundary [44], one can define a metric by extracting this pole. That is, pick a positive function r with a single zero at the boundary. The induced boundary metric is then given by $g_{(0)} = r^2 G|_{\partial M}$ where ∂M is the boundary of the manifold M . However, there is an obvious arbitrariness in this definition in that any other function $r' = e^w r$ with a single zero gives an equally valid boundary metric. Therefore, the metric on the boundary is defined up to a conformal transformation. This already indicates that the holographic dual should be a conformal theory, and is very similar to how in the eikonal regime of quantum gravity bulk time translations give rise to Lorentz boosts of the boundary theory.

On the other hand, infrared divergences break the symmetries of the bulk. To renormalise the theory we introduce a cut-off on the radial variable at $r = \epsilon$. One can then renormalise the action by adding covariant counter-terms which are evaluated at the cut-off $r = \epsilon$. When sending the cut-off to zero, the action should be finite. However, the presence of a logarithmic divergence gives rise to an anomaly when we perform a conformal transformation on $g_{(0)}$, $g'_{(0)} = e^{2\sigma} g_{(0)}$. This is a special kind of bulk diffeomorphism and so one would naively expect it to be a symmetry of the action. But it transforms as [70, 101]:

$$S_{\text{ren}}[e^{2\sigma} g_{(0)}] = S_{\text{ren}}[g_{(0)}] + \mathcal{A}[g_{(0)}, \sigma], \quad (4.4)$$

where the anomaly \mathcal{A} is a conformally invariant functional of the metric [70] and it precisely corresponds to the conformal anomalies found on the gauge theory side.

The fact that infrared divergences break bulk diffeomorphisms means that only diffeomorphisms that do not induce a Weyl rescaling on the boundary are true symmetries

of the theory. This implies that bulk solutions which are related by a diffeomorphism may have different dual stress-tensors when the diffeomorphism induces a conformal transformation on the boundary.

The discussion is qualitatively the same when one adds matter to the system. We discuss scalar fields but the discussion generalises straightforwardly to other kinds of matter. We study both the case when the gravitational background is fixed and the case when gravity is dynamical.

Let us summarise the results for the case of scalar fields in a fixed gravitational background (given by a metric of the form (4.1)). We look for solutions of massive scalar fields with mass $m^2 = (\Delta - d)\Delta$ that near the boundary have the form (in the co-ordinate system (4.1))

$$\begin{aligned} \Phi(x, r) = & r^{d-\Delta} (\phi_{(0)} + r^2 \phi_{(2)} + \dots + r^{2\Delta-d} \phi_{(2\Delta-d)} + \\ & + r^{2\Delta-d} \log r^2 \psi_{(2\Delta-d)}) + \mathcal{O}(r^{\Delta+1}). \end{aligned} \quad (4.5)$$

The logarithmic terms appears only when $2\Delta - d$ is an integer and we only consider this case in this chapter. We find that $\phi_{(n)}$, with $n < 2\Delta - d$, and $\psi_{(2\Delta-d)}$ are uniquely determined from the scalar field equation. This information is sufficient for a complete determination of the infrared divergences of the on-shell bulk action. In particular, the logarithmic term $\psi_{(2\Delta-d)}$ in (4.5) is directly related to matter conformal anomalies. These conformal anomalies were shown not to renormalise in [96]. We indeed find exact agreement with the computation in [96]. Adding counter-terms to cancel the infrared divergences we obtain the renormalised on-shell action. We stress that even in the case of a free massive scalar field in a fixed AdS background one needs counter-terms in order for the on-shell action to be finite (see (4.75)). The coefficient $\phi_{(2\Delta-d)}$ is left undetermined by the field equations. It is determined, however, by the expectation value of the dual operator $O(x)$. Differentiating the renormalised on-shell action one finds (up to terms contributing contact terms in the 2-point function)

$$\langle O(x) \rangle = (2\Delta - d)\phi_{(2\Delta-d)}(x). \quad (4.6)$$

This relation, with the precise proportionality coefficient, has first been derived in [81]. The value of the proportionality coefficient is crucial in order to obtain the correct normalisation of the 2-point function in standard AdS background [46].

In the case when the bulk geometry is dynamical we find that, for scalars that correspond to irrelevant operators, our perturbative treatment is consistent only if one considers single insertions of the irrelevant operator, i.e. the source is treated as an infinitesimal parameter, in agreement with the discussion in [125]. For scalars that correspond to marginal and relevant operators one can compute perturbatively the back-reaction of the scalars to the gravitational background. One can then regularise and renormalise as in the discussion of pure gravity or scalars in a fixed background. For illustrative purposes we analyse a simple example.

This chapter is organised as follows. In the next section we discuss the Dirichlet problem for AdS gravity and we obtain an asymptotic solution for a given boundary metric (up to six dimensions). In section 4.3 we use these solutions to obtain the infrared divergences of the on-shell gravitational action. After renormalising the on-shell action

by adding counter-terms, we compute the holographic stress-energy tensor. Section 4.4 is devoted to the study of the conformal transformation properties of the boundary stress-energy tensor. In section 4.5 we extend the analysis of sections 4.2 and 4.3 to include matter. In appendices C.1 and C.4 we give the explicit form of the solutions discussed in section 4.2 and section 4.5. Appendix C.2 contains the explicit form of the counter-terms discussed in section 4.3. In appendix C.3 we present a proof that the coefficient of the logarithmic term in the metric (present in even boundary dimensions) is proportional to the metric variation of the conformal anomaly.

4.2 Dirichlet boundary problem for AdS gravity

The Einstein-Hilbert action for a theory on a manifold M with boundary ∂M is given by⁵

$$S_{\text{gr}}[G] = \frac{1}{16\pi G_{\text{N}}} \left[\int_M d^{d+1}x \sqrt{G} (R[G] + 2\Lambda) - \int_{\partial M} d^d x \sqrt{\gamma} 2K \right], \quad (4.7)$$

where K is the trace of the second fundamental form (see (B.5)) and γ is the induced metric on the boundary. The boundary term is necessary in order to get an action which only depends on first derivatives of the metric [51], and it guarantees that the variational problem with Dirichlet boundary conditions is well-defined.

According to the prescription of [64, 125], the conformal field theory effective action is given by evaluating the on-shell action functional. The field specifying the boundary conditions for the metric is regarded as a source for the boundary operator. We therefore need to obtain solutions to Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2} R G_{\mu\nu} = \Lambda G_{\mu\nu}, \quad (4.8)$$

subject to appropriate Dirichlet boundary conditions.

As explained above, metrics $G_{\mu\nu}$ that satisfy (4.8) have a second order pole at infinity. Therefore, they do not induce a metric at infinity but only a conformal class. This is achieved by introducing a defining function r , i.e. a positive function in the interior of the manifold M that has a single zero and non-vanishing derivative at the boundary. Then one obtains a metric at the boundary by $g_{(0)} = r^2 G|_{\partial M}$ ⁶.

We are interested in solving (4.8) given a conformal structure at infinity. This can be achieved by working in the co-ordinate system (4.1) introduced by Fefferman and Graham [44]. The metric in (4.1) contains only even powers of r up to the order we are

⁵Our curvature conventions are as follows: $R_{ijk}{}^l = \partial_i \Gamma_{jk}{}^l + \Gamma_{ip}{}^l \Gamma_{jk}{}^p - (i \leftrightarrow j)$ and $R_{ij} = R_{ikj}{}^k$. We use these conventions the curvature of AdS comes out positive, but we will still use the terminology "space of constant negative curvature". Notice also that we take $\int d^{d+1}x = \int d^d x \int_0^\infty dr$ and the boundary is at $r = 0$ (in the co-ordinate system (4.1)). The minus sign in front of the trace of the second fundamental form is correlated with the choice of having $r = 0$ in the lower end of the radial integration.

⁶Throughout this chapter the metric $g_{(0)}$ is assumed to be non-degenerate. For studies of the AdS/CFT correspondence in cases where $g_{(0)}$ is degenerate we refer to [23, 110].

interested in [44] (see also [59, 57]). For this reason, it is convenient to use the variable $\rho = r^2$ [70],⁷

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu = l^2 \left(\frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j \right),$$

$$g(x, \rho) = g_{(0)} + \dots + \rho^{d/2} g_{(d)} + h_{(d)} \rho^{d/2} \log \rho + \dots, \quad (4.9)$$

where the logarithmic piece appears only for even d . The sub-index in the metric expansion (and in all other expansions that appear in this chapter) indicates the number of derivatives involved in that term, i.e. $g_{(2)}$ contains two derivatives, $g_{(4)}$ four derivatives, etc., as one can see from the explicit expressions given in appendix C.1. It follows that the perturbative expansion in ρ is also a low energy expansion. We set $l = 1$ from now on. One can easily reinstate the factors of l by dimensional analysis.

One can check that the curvature of G satisfies

$$R_{\kappa\lambda\mu\nu}[G] = (G_{\kappa\mu} G_{\lambda\nu} - G_{\kappa\nu} G_{\lambda\mu}) + \mathcal{O}(\rho). \quad (4.10)$$

In this sense the metric is asymptotically anti-de Sitter. The Dirichlet problem for Einstein metrics satisfying (4.10) exactly (i.e. not only to leading order in ρ) was solved in [102].

In the co-ordinate system (4.9), Einstein's equations read [70]

$$\begin{aligned} \rho [2g'' - 2g'g^{-1}g' + \text{Tr}(g^{-1}g')g'] + \text{Ric}(g) - (d-2)g' - \text{Tr}(g^{-1}g')g &= 0, \\ \nabla_i \text{Tr}(g^{-1}g') - \nabla^j g'_{ij} &= 0, \\ \text{Tr}(g^{-1}g'') - \frac{1}{2} \text{Tr}(g^{-1}g'g^{-1}g') &= 0 \end{aligned} \quad (4.11)$$

where differentiation with respect to ρ is denoted with a prime, ∇_i is the covariant derivative constructed from the metric g , and $\text{Ric}(g)$ is the Ricci tensor of g .

These equations are solved order by order in ρ . This is achieved by differentiating the equations with respect to ρ and then setting $\rho = 0$. For even d , this process would have broken down at order $d/2$ if the logarithm was not introduced in (4.9). $h_{(d)}$ is traceless, $\text{Tr} g_{(0)}^{-1} h_{(d)} = 0$, and covariantly conserved, $\nabla^i h_{(d)ij} = 0$. We show in appendix C.3 that $h_{(d)}$ is proportional to the metric variation of the corresponding conformal anomaly, i.e. it is proportional to the stress-energy tensor of the theory with action the conformal anomaly. In any dimension, only the trace of $g_{(d)}$ and its covariant divergence are determined. Here is where extra data from the CFT are needed: as we shall see, the undetermined part is specified by the expectation value of the dual stress-energy tensor.

We collect in appendix C.1 the results for $g_{(n)}$, $h_{(d)}$ as well as the results for the trace and divergence $g_{(d)}$. In dimension d the latter are the only constraints that equations (4.11) yield for $g_{(d)}$. From this information we can parametrise the indeterminacy by finding the most general $g_{(d)}$ that has the determined trace and divergence.

In $d = 2$ and $d = 4$ the equation for the coefficient $g_{(d)}$ has the form of a conservation law

$$\nabla^i g_{(d)ij} = \nabla^i A_{(d)ij}, \quad d = 2, 4, \quad (4.12)$$

⁷Greek indices, μ, ν, \dots are used for $d+1$ -dimensional indices, Latin ones, i, j, \dots for d -dimensional ones. To distinguish the curvatures of the various metrics introduced in (4.9) we will often use the notation $R_{ij}[g]$ to indicate that this is the Ricci tensor of the metric g , etc.

where $A_{(d)ij}$ is a symmetric tensor explicitly constructed from the coefficients $g_{(n)}$, $n < d$. The precise form of the tensor $A_{(d)ij}$ is given in appendix C.1 (eq.(C.4)). The integration of this equation obviously involves an ‘‘integration constant’’ $t_{ij}(x)$, a symmetric covariantly conserved tensor the precise form of which cannot be determined from Einstein’s equations.

In two dimensions, we get [102] (see also [15])

$$g_{(2)ij} = \frac{1}{2}(R g_{(0)ij} + t_{ij}), \quad (4.13)$$

where the symmetric tensor t_{ij} should satisfy

$$\nabla^i t_{ij} = 0, \quad \text{Tr } t = -R. \quad (4.14)$$

In four dimensions we obtain⁸

$$g_{(4)ij} = \frac{1}{8}g_{(0)ij} [(\text{Tr } g_{(2)})^2 - \text{Tr } g_{(2)}^2] + \frac{1}{2}(g_{(2)}^2)_{ij} - \frac{1}{4}g_{(2)ij} \text{Tr } g_{(2)} + t_{ij}. \quad (4.15)$$

The tensor t_{ij} satisfies

$$\nabla^i t_{ij} = 0, \quad \text{Tr } t = -\frac{1}{4}[(\text{Tr } g_{(2)})^2 - \text{Tr } g_{(2)}^2]. \quad (4.16)$$

In six dimensions the equation determining the coefficient $g_{(6)}$ is more subtle than the one in (4.12). It is given by

$$\nabla^i g_{(6)ij} = \nabla^i A_{(6)ij} + \frac{1}{6}\text{Tr } (g_{(4)} \nabla_j g_{(2)}), \quad (4.17)$$

where the tensor $A_{(6)ij}$ is given in (C.4). It contains a part which is antisymmetric in the indices i and j . Since $g_{(6)ij}$ is by definition a symmetric tensor the integration of equation (4.17) is not straightforward. Moreover, it is not obvious that the last term in (4.17) takes a form of divergence of some local tensor. Nevertheless, this is indeed the case as we now show. Let us define the tensor S_{ij} ,

$$S_{ij} = \nabla^2 C_{ij} - 2R^k{}_i{}^l{}_j C_{kl} + 4(g_{(2)}g_{(4)} - g_{(4)}g_{(2)})_{ij} + \frac{1}{10}(\nabla_i \nabla_j B - g_{(0)ij} \nabla^2 B) \\ + \frac{2}{5}g_{(2)ij} B + g_{(0)ij} \left(-\frac{2}{3}\text{Tr } g_{(2)}^3 - \frac{4}{15}(\text{Tr } g_{(2)})^3 + \frac{3}{5}\text{Tr } g_{(2)} \text{Tr } g_{(2)}^2 \right), \quad (4.18)$$

where

$$C_{ij} = (g_{(4)} - \frac{1}{2}g_{(2)}^2)_{ij} + \frac{1}{4}g_{(2)} \text{Tr } g_{(2)}_{ij} + \frac{1}{8}g_{(0)ij} B, \quad B = \text{Tr } g_{(2)}^2 - (\text{Tr } g_{(2)})^2.$$

The tensor S_{ij} is a local function of the Riemann tensor. Its divergence and trace read

$$\nabla^i S_{ij} = -4\text{Tr } (g_{(4)} \nabla_j g_{(2)})_{ij}, \quad \text{Tr } S = -8\text{Tr } (g_{(2)}g_{(4)})_{ij}. \quad (4.19)$$

⁸From now on we will suppress factors of $g_{(0)}$. For instance, $\text{Tr } g_{(2)}g_{(4)} = \text{Tr } [g_{(0)}^{-1}g_{(2)}g_{(0)}^{-1}g_{(4)}]$. Unless we explicitly mention the contrary, indices will be raised and lowered with the metric $g_{(0)}$, and all contractions will be made with this metric.

With the help of the tensor S_{ij} the equation (4.17) can be integrated in a way similar to the $d = 2, 4$ cases. One obtains

$$g_{(6)ij} = A_{(6)ij} - \frac{1}{24}S_{ij} + t_{ij} \quad . \quad (4.20)$$

Notice that tensor S_{ij} contains an antisymmetric part which cancels the antisymmetric part of the tensor $A_{(6)ij}$ so that $g_{(6)ij}$ and t_{ij} are symmetric tensors, as they should. The symmetric tensor t_{ij} satisfies

$$\nabla^i t_{ij} = 0 \quad , \quad \text{Tr } t = -\frac{1}{3}\left[\frac{1}{8}(\text{Tr } g_{(2)})^3 - \frac{3}{8}\text{Tr } g_{(2)}\text{Tr } g_{(2)}^2 + \frac{1}{2}\text{Tr } g_{(2)}^3 - \text{Tr } g_{(2)}g_{(4)}\right] \quad . \quad (4.21)$$

Notice that in all three cases, $d = 2, 4, 6$, the trace of t_{ij} is proportional to the holographic conformal anomaly. As we will see in the next section, the symmetric tensors t_{ij} are directly related to the expectation value of the boundary stress-energy tensor.

When d is odd the only constraint on the coefficient $g_{(d)ij}(x)$ is that it is conserved and traceless

$$\nabla^i g_{(d)ij} = 0 \quad , \quad \text{Tr } g_{(d)} = 0 \quad . \quad (4.22)$$

So that we may identify

$$g_{(d)ij} = t_{ij} \quad . \quad (4.23)$$

4.3 The holographic stress-energy tensor

We have seen in the previous section that given a conformal structure at infinity we can determine an asymptotic expansion of the metric up to order $\rho^{d/2}$. We will now show that this term is determined by the expectation value of the dual stress-energy tensor.

According to the AdS/CFT prescription, the expectation value of the boundary stress-energy tensor is determined by functionally differentiating the on-shell gravitational action with respect to the boundary metric. The on-shell gravitational action, however, diverges. To regulate the theory we restrict the bulk integral to the region $\rho \geq \epsilon$ and we evaluate the boundary term at $\rho = \epsilon$. The regulated action is given by

$$\begin{aligned} S_{\text{gr,reg}} &= \frac{1}{16\pi G_{\text{N}}} \left[\int_{\rho \geq \epsilon} d^{d+1}x \sqrt{G} (R[G] + 2\Lambda) - \int_{\rho=\epsilon} d^d x \sqrt{\gamma} 2K \right] = \quad (4.24) \\ &= \frac{1}{16\pi G_{\text{N}}} \int d^d x \left[\int_{\epsilon} d\rho \frac{d}{\rho^{d/2+1}} \sqrt{\det g(x, \rho)} \right. \\ &\quad \left. + \frac{1}{\rho^{d/2}} (-2d\sqrt{\det g(x, \rho)} + 4\rho\partial_{\rho}\sqrt{\det g(x, \rho)})|_{\rho=\epsilon} \right] , \end{aligned}$$

Evaluating (4.24) for the solution we obtained in the previous section we find that the divergences appears as $1/\epsilon^k$ poles plus a logarithmic divergence [70],

$$\begin{aligned} S_{\text{gr,reg}} &= \frac{l}{16\pi G_{\text{N}}} \int d^d x \sqrt{\det g_{(0)}} \left(\epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \dots + \epsilon^{-1} a_{(d-2)} \right. \\ &\quad \left. - \log \epsilon a_{(d)} \right) + \mathcal{O}(\epsilon^0), \quad (4.25) \end{aligned}$$

where the coefficients $a_{(n)}$ are local covariant expressions of the metric $g_{(0)}$ and its curvature tensor. We give the explicit expressions, up to the order we are interested in, in appendix C.

We now obtain the renormalised action by subtracting the divergent terms, and then removing the regulator,

$$S_{\text{gr,ren}}[g_{(0)}] = \lim_{\epsilon \rightarrow 0} \frac{1}{16\pi G_{\text{N}}} \left[S_{\text{gr,reg}} - \int d^d x \sqrt{\det g_{(0)}} \left(\epsilon^{-d/2} a_{(0)} + \epsilon^{-d/2+1} a_{(2)} + \dots \right. \right. \\ \left. \left. + \epsilon^{-1} a_{(d-2)} - \log \epsilon a_{(d)} \right) \right]. \quad (4.26)$$

The expectation value of the stress-energy tensor of the dual theory is given by

$$\langle T_{ij} \rangle = \frac{2}{\sqrt{\det \gamma_{(0)}}} \frac{\partial S_{\text{gr,ren}}}{\partial g_{(0)}^{ij}} = \lim_{\epsilon \rightarrow 0} \frac{2}{\sqrt{\det g(x, \epsilon)}} \frac{\partial S_{\text{gr,ren}}}{\partial g^{ij}(x, \epsilon)} = \lim_{\epsilon \rightarrow 0} \left(\frac{1}{\epsilon^{d/2-1}} T_{ij}[\gamma] \right), \quad (4.27)$$

where $T_{ij}[\gamma]$ is the stress-energy tensor of the theory at $\rho = \epsilon$ described by the action in (4.26) but before the limit $\epsilon \rightarrow 0$ is taken ($\gamma_{ij} = 1/\epsilon g_{ij}(x, \epsilon)$ is the induced metric at $\rho = \epsilon$). Notice that the asymptotic expansion of the metric only allows for the determination of the divergences of the on-shell action. We can still obtain, however, a formula for $\langle T_{ij} \rangle$ in terms of $g_{(n)}$ since, as (4.27) shows, we only need to know the first $\epsilon^{d/2-1}$ orders in the expansion of $T_{ij}[\gamma]$.

The stress-energy tensor $T_{ij}[\gamma]$ contains two contributions,

$$T_{ij}[\gamma] = T_{ij}^{\text{reg}} + T_{ij}^{\text{ct}}, \quad (4.28)$$

T_{ij}^{reg} comes from the regulated action in (4.24) and T_{ij}^{ct} is due to the counter-terms. The first contribution is equal to

$$T_{ij}^{\text{reg}}[\gamma] = -\frac{1}{8\pi G_{\text{N}}} (K_{ij} - K\gamma_{ij}) \\ = -\frac{1}{8\pi G_{\text{N}}} (-\partial_\epsilon g_{ij}(x, \epsilon) + g_{ij}(x, \epsilon) \text{Tr} [g^{-1}(x, \epsilon) \partial_\epsilon g(x, \epsilon)]) \\ + \frac{1-d}{\epsilon} g_{ij}(x, \epsilon). \quad (4.29)$$

The contribution due to counter-terms can be obtained from the results in appendix C.2. It is given by

$$T_{ij}^{\text{ct}} = -\frac{1}{8\pi G_{\text{N}}} \left((d-1)\gamma_{ij} + \frac{1}{(d-2)} (R_{ij} - \frac{1}{2} R\gamma_{ij}) + \right. \\ \left. -\frac{1}{(d-4)(d-2)^2} [-\nabla^2 R_{ij} + 2R_{ikjl}R^{kl} + \frac{d-2}{2(d-1)} \nabla_i \nabla_j R - \frac{d}{2(d-1)} RR_{ij} \right. \\ \left. -\frac{1}{2} \gamma_{ij} (R_{kl}R^{kl} - \frac{d}{4(d-1)} R^2 - \frac{1}{d-1} \nabla^2 R) \right] - T_{ij}^a \log \epsilon \Big), \quad (4.30)$$

where T_{ij}^a is the stress-energy tensor of the action $\int d^d x \sqrt{\det \gamma} a_{(d)}$. As is shown in Appendix C.3, T_{ij}^a is proportional to the tensor $h_{(d)ij}$ appearing in the expansion (4.9).

The stress tensor $T_{ij}[g_{(0)}]$ is covariantly conserved with respect to the metric $g_{(0)ij}$. To see this, notice that each of T_{ij}^{reg} and T_{ij}^{ct} is separately covariantly conserved with respect to the induced metric γ_{ij} at $\rho = \epsilon$: for T_{ij}^{reg} one can check this by using the second equation in (4.11), for T_{ij}^{ct} this follows from the fact that it was obtained by varying a local covariant counter-term. Since all divergences cancel in (4.27), we obtain that the finite part in (4.27) is conserved with respect to the metric $g_{(0)ij}$.

We are now ready to calculate T_{ij} . By construction (and we will verify this below) the divergent pieces cancel between T^{reg} and T^{ct} .

4.3.1 $d = 2$

In two dimensions we obtain

$$\langle T_{ij} \rangle = \frac{l}{16\pi G_{\text{N}}} t_{ij}, \quad (4.31)$$

where we have used (4.13) and (4.14) and the fact that $T_{ij}^a = 0$ since $\int R$ is a topological invariant (and reinstated the factor of l). As promised, t_{ij} is directly related to the boundary stress-energy tensor. Taking the trace we obtain

$$\langle T_i^i \rangle = -\frac{c}{24\pi} R, \quad (4.32)$$

where $c = 3l/2G_{\text{N}}$, which is the correct conformal anomaly [24].

Using our results, one can immediately obtain the stress-energy tensor of the boundary theory associated with a given solution G of the three dimensional Einstein equations: one needs to write the metric in the co-ordinate system (4.9) and then use the formula

$$\langle T_{ij} \rangle = \frac{2l}{16\pi G_{\text{N}}} (g_{(2)ij} - g_{(0)ij} \text{Tr } g_{(2)}). \quad (4.33)$$

From the gravitational point of view this is the quasi-local stress energy tensor associated with the solution G .

4.3.2 $d = 4$

To obtain T_{ij} we first need to rewrite the expressions in T^{ct} in terms of $\gamma_{(0)}$. This can be done with the help of the relation

$$\begin{aligned} R_{ij}[\gamma] &= R_{ij}[\gamma_{(0)}] + \frac{1}{4}\epsilon \left(2R_{ik}R^k{}_j - 2R_{ikjl}R^{kl} - \frac{1}{3}\nabla_i\nabla_j R + \nabla^2 R_{ij} - \frac{1}{6}\nabla^2 R g_{(0)ij} \right) \\ &+ \mathcal{O}(\epsilon^2). \end{aligned} \quad (4.34)$$

After some algebra one obtains,

$$\begin{aligned} \langle T_{ij}[g_{(0)}] \rangle &= -\frac{1}{8\pi G_{\text{N}}} \lim_{\epsilon \rightarrow 0} \left[\frac{1}{\epsilon} (-g_{(2)ij} + g_{(0)ij} \text{Tr } g_{(2)} + \frac{1}{2}R_{ij} - \frac{1}{4}g_{(0)ij}R) \right. \\ &\quad \left. + \log \epsilon (-2h_{(4)ij} - T_{ij}^a) - 2g_{(4)ij} - h_{(4)ij} - g_{(2)ij} \text{Tr } g_{(2)} - \frac{1}{2}g_{(0)ij} \text{Tr } g_{(2)}^2 \right] \end{aligned}$$

$$\begin{aligned}
& +\frac{1}{8}(R_{ik}R^k{}_j - 2R_{ikjl}R^{kl} - \frac{1}{3}\nabla_i\nabla_j R + \nabla^2 R_{ij} - \frac{1}{6}\nabla^2 Rg_{(0)ij}) \\
& -\frac{1}{4}g_{(2)ij}R + \frac{1}{8}g_{(0)ij}(R_{kl}R^{kl} - \frac{1}{6}R^2) \Big]. \tag{4.35}
\end{aligned}$$

Using the explicit expression for $g_{(2)}$ and $h_{(4)}$ given in (C.1) and (C.6) one finds that both the $1/\epsilon$ pole and the logarithmic divergence cancel. Notice that had we not subtracted the logarithmic divergence from the action, the resulting stress-energy tensor would have been singular in the limit $\epsilon \rightarrow 0$.

Using (4.15) and (4.16) and after some algebra we obtain

$$\langle T_{ij} \rangle = -\frac{1}{8\pi G_N}[-2t_{ij} - 3h_{(4)}]. \tag{4.36}$$

Taking the trace we get

$$\langle T^i{}_i \rangle = \frac{1}{16\pi G_N}(-2a_{(4)}), \tag{4.37}$$

which is the correct conformal anomaly [70].

Notice that since $h_{(4)ij} = -\frac{1}{2}T_{ij}^a$ the contribution in the boundary stress-energy tensor proportional to $h_{(4)ij}$ is scheme-dependent. Adding a local finite counter-term proportional to the trace anomaly will change the coefficient of this term. One may remove this contribution from the boundary stress energy tensor by a choice of scheme.

Finally, one can obtain the energy-momentum tensor of the boundary theory for a given solution G of the five dimensional Einstein equations with negative cosmological constant. It is given by

$$\langle T_{ij} \rangle = \frac{4}{16\pi G_N}[g_{(4)ij} - \frac{1}{8}g_{(0)ij}[(\text{Tr } g_{(2)})^2 - \text{Tr } g_{(2)}^2] - \frac{1}{2}(g_{(2)}^2)_{ij} + \frac{1}{4}g_{(2)ij}\text{Tr } g_{(2)}], \tag{4.38}$$

where we have omitted the scheme-dependent $h_{(4)}$ -terms. From the gravitational point of view this is the quasi-local stress energy tensor associated with the solution G .

4.3.3 $d = 6$

The calculation of the boundary stress tensor in the $d = 6$ case goes along the same lines as in $d = 2$ and $d = 4$ cases although it is technically involved. Up to a local traceless covariantly conserved term (proportional to $h_{(6)}$) the result is

$$\langle T_{ij} \rangle = \frac{3}{8\pi G_N}(g_{(6)ij} - A_{(6)ij} + \frac{1}{24}S_{ij}) \ . \tag{4.39}$$

where $A_{(6)ij}$ is given in (C.4) and S_{ij} in (4.18). It is covariantly conserved and has the correct trace

$$\langle T^i{}_i \rangle = \frac{1}{8\pi G_N}(-a_{(6)}) \ , \tag{4.40}$$

reproducing correctly the conformal anomaly in six dimensions [70].

Given an asymptotically AdS solution in six dimensions equation (4.39) yields the quasi-local stress-energy tensor associated with it.

4.3.4 $d = 2k + 1$

In this case one can check that the counter-terms only cancel infinities. Evaluating the finite part we get

$$\langle T_{ij} \rangle = \frac{d}{16\pi G_N} g_{(d)ij}, \quad (4.41)$$

where $g_{(d)ij}$ can be identified with a traceless covariantly conserved tensor t_{ij} . In odd boundary dimensions there are no gravitational conformal anomalies, and indeed (4.41) is traceless. As in all previous cases, one can also read (4.41) as giving the quasi-local stress-energy tensor associated with a given solution of Einstein's equations.

4.3.5 Conformally flat bulk metrics

In this subsection we discuss a special case where the bulk metric can be determined to all orders given only a boundary metric. It was shown in [102] that, given a conformally flat boundary metric, equations (4.11) can be integrated to all orders if the bulk Weyl tensor vanishes⁹. We show that the extra condition in the bulk metric singles out a specific vacuum of the CFT.

The solution obtained in [102] is given by

$$g(x, \rho) = g_{(0)}(x) + g_{(2)}(x)\rho + g_{(4)}(x)\rho^2, \quad g_{(4)} = \frac{1}{4}(g_{(2)})^2, \quad (4.42)$$

where $g_{(2)}$ is given in (C.1) (we consider $d > 2$), and all other coefficients $g_{(n)}$, $n > 4$ vanish. Since $g_{(4)}$ and $g_{(6)}$ are now known, one can obtain a local formula for the dual stress-energy tensor in terms of the curvature by using (4.15) and (4.20).

In $d = 4$, using (4.15) and $g_{(4)} = \frac{1}{4}(g_{(2)})^2$, one obtains

$$t_{ij} = t_{ij}^{\text{cf}} \equiv -\frac{1}{4}(g_{(2)})^2_{ij} + \frac{1}{4}g_{(2)ij}\text{Tr } g_{(2)} - \frac{1}{8}g_{(0)ij}[(\text{Tr } g_{(2)})^2 - \text{Tr } g_{(2)}^2]. \quad (4.43)$$

It is easy to check that trace of t_{ij}^{cf} reproduces (4.16). Furthermore, by virtue of the Bianchi identities, one can show that t_{ij}^{cf} is covariantly conserved. It is well-known that the stress-energy tensor of a quantum field theory on a conformally flat space-time is a local function of the curvature tensor (see for example the book by Birrell and Davies, [20]). Our equation (4.43) reproduces the corresponding expression given in [20].

In $d = 6$, using (4.20) and $g_{(6)} = 0$ we find

$$t_{ij} = t_{ij}^{\text{cf}} \equiv \left[\frac{1}{4}g_{(2)}^3 - \frac{1}{4}g_{(2)}^2\text{Tr } g_{(2)} + \frac{1}{8}g_{(2)}(\text{Tr } g_{(2)})^2 - \frac{1}{8}g_{(2)}\text{Tr } g_{(2)} \right. \\ \left. + g_{(0)}\left(\frac{1}{8}\text{Tr } g_{(2)}\text{Tr } g_{(2)}^2 - \frac{1}{12}\text{Tr } g_{(2)}^3 - \frac{1}{24}(\text{Tr } g_{(2)})^3\right) \right]_{ij}. \quad (4.44)$$

⁹In [102] it was proven that if the bulk metric satisfies Einstein's equations and it has a vanishing Weyl tensor, then the corresponding boundary metric has to be conformally flat. The converse is not necessarily true: one can have Einstein metrics with non-vanishing Weyl tensor which induce a conformally flat metric in the boundary.

One can verify that the trace of t_{ij}^{cf} reproduces (4.21) (taking into account that $g_{(4)} = \frac{1}{4}g_{(2)}^2$ and that t_{ij}^{cf} is covariantly conserved (by virtue of the Bianchi identities)).

Following the analysis in the previous subsections we obtain

$$\langle T_{ij} \rangle = \frac{d}{16\pi G_{\text{N}}} t_{ij}^{\text{cf}}. \quad (4.45)$$

So, we explicitly see that the global condition we imposed on the bulk metric implies that we have picked a particular vacuum in the conformal field theory.

Note that the tensors t_{ij}^{cf} in (4.43), (4.44) are local polynomial functions of the Ricci scalar and the Ricci tensor (but not of the Riemann tensor) of the metric $g_{(0)ij}$. It is perhaps an expected but still a surprising result that in conformally flat backgrounds the anomalous stress tensor is a local function of the curvature.

4.4 Conformal transformation properties of the stress-energy tensor

In this section we discuss the conformal transformation properties of the stress-energy tensor. These can be obtained by noting [75] that conformal transformations in the boundary originate from specific diffeomorphisms that preserve the form of the metric (4.9). Under these diffeomorphisms $g_{ij}(x, \rho)$ transforms infinitesimally as [75]

$$\delta g_{ij}(x, \rho) = 2\sigma(1 - \rho\partial_\rho) g_{ij}(x, \rho) + \nabla_i a_j(x, \rho) + \nabla_j a_i(x, \rho), \quad (4.46)$$

where $a_j(x, \rho)$ is obtained from the equation

$$a^i(x, \rho) = \frac{1}{2} \int_0^\rho d\rho' g^{ij}(x, \rho') \partial_j \sigma(x). \quad (4.47)$$

This can be integrated perturbatively in ρ ,

$$a^i(x, \rho) = \sum_{k=1} a_{(k)}^i \rho^k. \quad (4.48)$$

We will need the first two terms in this expansion,

$$a_{(1)}^i = \frac{1}{2} \partial^i \sigma, \quad a_{(2)}^i = -\frac{1}{4} g_{(2)}^{ij} \partial_j \sigma. \quad (4.49)$$

We can now obtain the way the $g_{(n)}$'s transform under conformal transformations [75]

$$\begin{aligned} \delta g_{(0)ij} &= 2\sigma g_{(0)ij}, \\ \delta g_{(2)ij} &= \nabla_i a_{(1)j} + \nabla_j a_{(1)i}, \\ \delta g_{(3)ij} &= -\sigma g_{(3)ij}, \\ \delta g_{(4)ij} &= -2\sigma(g_{(4)} + h_{(4)}) + a_{(1)}^k \nabla_k g_{(2)ij} + \nabla_i a_{(2)j} + \nabla_j a_{(2)i} \\ &\quad + g_{(2)ik} \nabla_j a_{(1)}^k + g_{(2)jk} \nabla_i a_{(1)}^k, \\ \delta g_{(5)ij} &= -3\sigma g_{(3)ij}, \end{aligned} \quad (4.50)$$

where the term $h_{(4)}$ in $g_{(4)}$ is only present when $d = 4$. One can check from the explicit expressions for $g_{(2)}$ and $g_{(4)}$ in (C.1) that they indeed transform as (4.50). An alternative way to derive the transformation rules above is to start from (C.1) and perform a conformal variation. In [75] the variations (4.50) were integrated leading to (C.1) up to conformally invariant terms.

Equipped with these results and the explicit form of the energy-momentum tensors, we can now easily calculate how the quantum stress-energy tensor transforms under conformal transformations. We use the term “quantum stress-energy tensor” because it incorporates the conformal anomaly. In the literature such transformation rules were obtained [27] by first integrating the conformal anomaly to an effective action. This effective action is a functional of the initial metric g and of the conformal factor σ . It can be shown that the difference between the stress-energy tensor of the theory on the manifold with metric $ge^{2\sigma}$ and the one on the manifold with metric g is given by the stress-energy tensor derived by varying the effective action with respect to g .

In any dimension the stress-energy tensor transforms *classically* under conformal transformations as

$$\delta\langle T_{\mu\nu}\rangle = -(d-2)\sigma\langle T_{\mu\nu}\rangle. \quad (4.51)$$

This transformation law is modified by the quantum conformal anomaly. In odd dimensions, where there is no conformal anomaly, the classical transformation rule (4.51) holds also at the quantum level. Indeed, for odd d , and by using (4.41) and (4.50), one easily verifies that the holographic stress-energy tensor transforms correctly.

In even dimensions, the transformation (4.51) is modified. In $d = 2$, it is well-known that one gets an extra contribution proportional to the central charge. Indeed, using (4.33) and the formulae above we obtain

$$\delta\langle T_{ij}\rangle = \frac{l}{8\pi G_N}(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) = \frac{c}{12}(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma), \quad (4.52)$$

which is the correct transformation rule.

In $d = 4$ we obtain,

$$\begin{aligned} \delta\langle T_{ij}\rangle &= -2\sigma\langle T_{ij}\rangle \\ &+ \frac{1}{4\pi G_N}\left(-2\sigma h_{(4)} + \frac{1}{4}\nabla^k\sigma[\nabla_k R_{ij} - \frac{1}{2}(\nabla_i R_{jk} + \nabla_j R_{ik}) - \frac{1}{6}\nabla_k R g_{(0)ij}] \right. \\ &+ \frac{1}{48}(\nabla_i\sigma\nabla_j R + \nabla_i\sigma\nabla_j R) + \frac{1}{12}R(\nabla_i\nabla_j\sigma - g_{(0)ij}\nabla^2\sigma) \\ &\left. + \frac{1}{8}[R_{ij}\nabla^2\sigma - (R_{ik}\nabla^k\nabla_j\sigma + R_{jk}\nabla^k\nabla_i\sigma) + g_{(0)ij}R_{kl}\nabla^k\nabla^l\sigma]\right). \quad (4.53) \end{aligned}$$

The only other result known to us is the result in [27], where they computed the finite conformal transformation of the stress-energy tensor but for a conformally flat metric $g_{(0)}$. For conformally flat backgrounds, $h_{(4)}$ vanishes because it is the metric variation of a topological invariant. The terms proportional to a single derivative of σ vanish by virtue of Bianchi identities and the fact that the Weyl tensor vanishes for conformally

flat metrics. The remaining terms, which only contain second derivatives of σ , can be shown to coincide with the infinitesimal version of (4.23) in [27].

One can obtain the conformal transformation of the stress-energy tensor in $d = 6$ in a similar fashion but we shall not present this result here.

4.5 Matter

In the previous sections we examined how space-time is reconstructed (to leading order) holographically out of CFT data. In this section we wish to examine how field theory describing matter on this space-time is encoded in the CFT. We will discuss scalar fields but the techniques are readily applicable to other kinds of matter.

The method we will use is the same as in the case of pure gravity, i.e. we will start by specifying the sources that are turned on, find how far we can go with only this information and then input more CFT data. We will find the same pattern: knowledge of the sources allows only for determination of the divergent part of the action. The leading finite part (which depends on global issues and/or the signature of space-time) is determined by the expectation value of the dual operator. We would like to stress that in the approach we follow, i.e. regularise, subtract all infinities by adding counter-terms and finally remove the regulator to obtain the renormalised action, all normalisations of the physical correlation functions are fixed and are consistent with Ward identities.

Other papers that discuss similar issues include [8, 93, 92, 111].

In order to couple gravity to matter, one has to solve the coupled system of Einstein's equations and the matter field equations. This is non-trivial, as in general it is hard to solve them exactly. In particular, it is not enough to have a solution $G_{\mu\nu}$ of Einstein's equations given some matter fields, denoted collectively by $\Phi(x)$, which enter Einstein's equations through the stress-energy tensor $T_{\mu\nu}$. One also has to ensure that the fields $\Phi(x)$ remain a solution of the matter field equations for the metric $G_{\mu\nu}$ with back-reaction. A simple example where this is the case is the shock-wave solution discussed in chapter 2. This solution is an exact solution of Einstein's equations with stress-energy tensor $T_{vv} = -p \delta(v) \delta^{(d-2)}(x)$. Now a straightforward analysis of the geodesics in the shock-wave metric, (2.19), shows that the null geodesic $v = 0$, $x^i = 0$ remains a null geodesic in the shock-wave metric. The reason is that the shock-wave metric still has an isometry along the u -direction. So, the stress-energy tensor does not change and the shock-wave solution solves both the Einstein and the matter field equations exactly: there is no gravitational self-interaction.

In general, however, it is hard to find exact solutions and one takes a perturbative approach, assuming that the matter content perturbs the space-time only slightly. So, as long as the geometry is not too violently modified, one can set up a perturbative expansion where the expansion parameter is the Planck length divided by the typical length scale set by matter. So, one usually neglects the second-order back-reaction which is produced by the changes in the matter field equations induced by the first order back-reaction. This is the approach we will pursue here.

In addition, since we look for perturbative solutions of Einstein's equations near the boundary, also the matter system should have perturbative solutions near the boundary. In other words, we need a perturbative expansion of the stress-energy tensor in r . The

existence of perturbative solutions of Einstein's equations sets constraints on the allowed behaviour of the stress-energy tensor near the boundary. For the scalar fields of mass m that we will study in the next section, this implies $m^2 \leq 0$. With these constraints, it is easy to check that the leading behaviour of the stress-energy tensor does not change when we take into account the back-reaction on the metric. This is because, from the CFT point of view, turning on a source $\phi_{(0)}$ or giving a non-vanishing expectation value to the operator $O(x)$ of dimension Δ to which the source couples only changes the expectation value of the stress-energy tensor, but not the metric $g_{(0)}$. In other words, we still have a genuine Dirichlet problem in the bulk and only normalisable solutions change. It is possible to find the general expansion of the stress-energy tensor in r up to the desired order and including an arbitrary number of back-reaction steps, but in most cases the second-order back-reaction effects do not affect the bulk metric to the order we are interested in.

4.5.1 Coupling gravity to matter

In this section we make some preliminary remarks concerning the existence of solutions of Einstein's equations coupled to matter. We do this very generally, without assuming any specific matter model.

The local analysis in the previous sections revealed that undeterminacies in the bulk metric in asymptotically AdS spaces are directly related to information about expectation values of operators in the CFT. For future reference, let us write the three components of Einstein's equations as follows:

$$\begin{aligned} E_{ij} &= 0 \\ E_{ri} &= 0 \\ E_{rr} &= 0 \end{aligned} \tag{4.54}$$

where

$$E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R - \Lambda G_{\mu\nu} + 8\pi G_{\text{N}} T_{\mu\nu}. \tag{4.55}$$

The three components of (4.54) are of course the three components of (4.11) coupled to arbitrary matter.

Now an essential fact in our analysis of the previous sections was that the (ij) -component of Einstein's equations (4.54) left undetermined the metric coefficient $g_{(d)}$. Its trace was determined by the third of (4.54), and the second of (4.54) gave additional information about the traceless part of $g_{(d)}$. This seems to be at odds with the fact that Einstein's equations have some degeneracy related to co-ordinate invariance, and the first and third of (4.54) together with the Bianchi identities are generally sufficient to solve the second one. We will see that this general expectation is only true up to "integration constants". It is interesting to study this in some detail, as the information missing from the second of (4.54) was exactly the information about the dual stress-energy tensor. Indeed, as we shall now see, one can prove that under certain constraints the first and third of (4.54) are enough to satisfy the second of (4.54) *up to a certain coefficient*. This coefficient is exactly the one that specifies the dual stress-energy tensor.

The same is true for the third of (4.54): a certain integration constant has to be set to zero, and this in turn gives the right value for the conformal anomaly. Our only restrictions are that (4.54) has perturbative solutions in r , and that we work with the lowest-order supergravity action without α' -corrections.

In this section we work in the r -co-ordinate system (4.1). It is convenient to first work out the Ricci tensor in (4.54):

$$\begin{aligned}
R_{ij} &= R_{ij}(g) + \frac{d}{r^2} g_{ij} - \frac{d-1}{2r} g'_{ij} + \frac{1}{2} g''_{ij} - \frac{1}{2} (g' g^{-1} g')_{ij} \\
&\quad + \frac{1}{4} g'_{ij} \text{Tr}(g^{-1} g') - \frac{1}{2r} g_{ij} \text{Tr}(g^{-1} g') \\
R_{ir} &= \frac{1}{2} (g^{-1})^{jk} (\nabla_i g'_{jk} - \nabla_k g'_{ij}) \\
R_{rr} &= \frac{d}{r^2} - \frac{1}{2r} \text{Tr}(g^{-1} g') + \frac{1}{2} \text{Tr}(g^{-1} g'') - \frac{1}{4} \text{Tr}(g^{-1} g')^2. \tag{4.56}
\end{aligned}$$

We see from (4.56) and (4.54) that, for the existence of perturbative solutions, the stress-energy tensor is not allowed to diverge worse than $1/r$. Thus, we consider the perturbative expansion: $T_{\mu\nu} = \frac{1}{r} T_{(-1)\mu\nu} + T_{(0)\mu\nu} + \dots$. For T_{ir} we have the stronger requirement $T_{(-1)ir} = 0$. In later sections we will make some comments on stress-energy tensors that have a more violent decay near the boundary. The stress-energy tensor can also contain logarithmic contributions, but usually these appear at higher order and we will not consider them here.

In the co-ordinate system (4.1), the Bianchi identities take the following form:

$$[(d-1) - \frac{r}{2} \text{Tr}(g^{-1} g')] E_{ir} - r E'_{ir} = r \nabla^k E_{ik} \tag{4.57}$$

$$\begin{aligned}
[(d-2) - \frac{r}{2} \text{Tr}(g^{-1} g')] r E_{rr} - r^2 E'_{rr} &= r \text{Tr}(g^{-1} E) - \frac{r^2}{2} \text{Tr}(g^{-1} g' g^{-1} E) + \\
&\quad + r^2 \nabla^k E_{rk}, \tag{4.58}
\end{aligned}$$

Substituting our ansatz for the metric, (4.9), for the first Bianchi identity at lowest order we get:

$$(d-1) E_{ir}|_{r=0} = E_{(-1)ij}. \tag{4.59}$$

Now if the first Einstein equation is satisfied at lowest order, $E_{(-1)ij} = 0$, then so is the second, $E_{(0)ir} = 0$.

Now we can use induction to see whether, if $E_{ij} = 0$ to all orders, $E_{ir} = 0$ is true to all orders as well. We take successive derivatives of (4.58), which at order n gives the expression:

$$\sum_{k=0}^{n+1} a_k^{n+1}(r) E_{ir}^{(k)}|_{r=0} = 0, \tag{4.60}$$

a being some coefficient with the property $a_{n+1}^{n+1}(r=0) = 0$. The vanishing of (4.60) would be enough to ensure $E_{ir} = 0$ at each order. However, if some a_n^{n+1} vanishes, the

equation cannot be solved and so at that order we may need to introduce logarithmic terms. This happens exactly for $n + 1 = d$. So, the perturbative analysis reveals that $E_{ij} = 0$ ensures $E_{ir} = 0$ only up to order $d - 1$. Let us analyse this in some more detail.

Assuming $E_{ij} = 0$ to all orders, (4.58) reduces to

$$[(d - 1) - \frac{r}{2} \text{Tr}(g^{-1}g')]E_{ir} - rE'_{ir} = 0. \quad (4.61)$$

This we can integrate exactly, getting

$$E_{ir} = c_i r^{d-1} e^{-H(r)}, \quad (4.62)$$

where $H(r) = \frac{1}{2} \int dr \text{Tr}(g^{-1}g')$ and therefore it has the same regular power expansion as g . We thus see that, in general, we need to impose the additional constraint $c_i = 0$ for $E_{ir} = 0$ to be true. This is equivalent to setting

$$c_i = E'_{ir}{}^{(d-1)}|_{r=0} = 0. \quad (4.63)$$

The fact that (4.63) is met at order $d - 1$ is not accidental. This is exactly the same behaviour we encountered in the vacuum case. So, it is true that the first of Einstein's equations together with the Bianchi equation imply the second Einstein equation, only if (4.63) is satisfied. The latter condition in turn implies that the d -th derivative of g is not specified by the first Einstein equations and has to be imposed additionally. Thus, the second of Einstein's equations gives us information about the traceless part of $g_{(d)}$.

The same analysis can be done for the second Bianchi identity (4.58). We get

$$\begin{aligned} [(d - 2) - \frac{r}{2} \text{Tr}(g^{-1}g')]rE_{rr} - r^2E'_{rr} &= r\text{Tr}(g^{-1}E) - \frac{r^2}{2} \text{Tr}(g^{-1}g'g^{-1}E) + \\ &+ r^2\nabla^k E_{rk}. \end{aligned} \quad (4.64)$$

Now, assuming $E_{ij} = 0$ and $c_i = 0$ implies $E_{ir} = 0$ by the previous argument, and this gives an equation for E_{rr} with the following exact solution

$$E_{rr} = D r^{d-2} e^{-H(r)}. \quad (4.65)$$

Thus, we also have to impose $D = 0$, which gives the condition

$$D = E'_{rr}{}^{(d-2)}|_{r=0} = 0. \quad (4.66)$$

This ensures $E_{rr} = 0$ to all orders and imposes a further constraint on the d -th derivative of g , which now has to satisfy (4.63) and (4.66): in the vacuum case, the latter condition determines the trace of $g_{(d)}$.

To summarise, we have found that $E_{ij} = 0$ and the Bianchi identities are not enough to have a solution of Einstein's equations. One needs to set to zero two additional integration constants, and these determine (part of) the coefficient $g_{(d)}$ in the expansion of the metric. Notice, however, that setting these integration constants to zero only ensures the existence of a solution of Einstein's equations, but does not necessarily specify all the coefficients of the metric uniquely. In fact, as we saw in the previous sections, the traceless part of $g_{(d)}$ is still undetermined.

It of course remains to be shown that the first of (4.54) indeed has solutions to all orders given an arbitrary boundary condition $g_{(0)}$. For maximally symmetric spaces this was done in [102].

4.5.2 Dirichlet boundary problem for scalar fields in a fixed background

In this section we consider scalars on a fixed gravitational background. This is taken to be of the generic form (4.9). In most of the literature the fixed metric was taken to be that of standard AdS, but with not much more effort one can consider the general case.

The action for a massive scalar is given by

$$S_M = \frac{1}{2} \int d^{d+1}x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2), \quad (4.67)$$

where $G_{\mu\nu}$ has an expansion of the form (4.9).

We take the scalar field Φ to have an expansion of the form

$$\Phi(x, \rho) = \rho^{(d-\Delta)/2} \phi(x, \rho), \quad \phi(x, \rho) = \phi_{(0)} + \phi_{(2)}\rho + \dots, \quad (4.68)$$

where Δ is the conformal dimension of the dual operator. We take the dimension Δ to be quantised as $\Delta = \frac{d}{2} + k, k = 0, 1, \dots$. This is often the case for operators of protected dimension. For the case of scalars that correspond to operators of dimensions $\frac{d}{2} - 1 \leq \Delta < \frac{d}{2}$ we refer to [81]. Inserting (4.68) in the field equation,

$$(-\square_G + m^2)\Phi = 0, \quad (4.69)$$

where $\square_G \Phi = \frac{1}{\sqrt{G}} \partial_\mu (\sqrt{G} G^{\mu\nu} \partial_\nu \Phi)$, we obtain that the mass m^2 and the conformal dimension Δ are related as $m^2 = (\Delta - d)\Delta$, as explained in the introduction, see (1.52). ϕ satisfies

$$[-(d - \Delta)\partial_\rho \log g \phi + 2(2\Delta - d - 2)\partial_\rho \phi - \square_g \phi] + \rho[-2\partial_\rho \log g \partial_\rho \phi - 4\partial_\rho^2 \phi] = 0. \quad (4.70)$$

Given $\phi_{(0)}$ one can determine recursively $\phi_{(n)}, n > 0$. This is achieved by differentiating (4.70) and setting ρ equal to zero. We give the result for the first couple of orders in appendix C.4. This process breaks down at order $\Delta - d/2$ (provided this is an integer, which we assume throughout this section) because the coefficient of $\phi_{(2\Delta-d)}$ (the field to be determined) becomes zero. This is exactly analogous to the situation encountered for even d in the gravitational sector. Exactly the same way as there, we introduce at this order a logarithmic term, i.e. the expansion of Φ now reads,

$$\Phi = \rho^{(d-\Delta)/2} (\phi_{(0)} + \rho\phi_{(2)} + \dots) + \rho^{\Delta/2} (\phi_{(2\Delta-d)} + \log \rho \psi_{(2\Delta-d)} + \dots). \quad (4.71)$$

The equation (4.70) now determines all terms up to $\phi_{(2\Delta-d-2)}$, the coefficient of the logarithmic term $\psi_{(2\Delta-d)}$, but leaves undetermined $\phi_{(2\Delta-d)}$. This is analogous to the situation discussed in section 4.2, where the term $g_{(d)}$ was undetermined. It is well known [12, 13, 81] that precisely at order $\rho^{\Delta/2}$ one finds the expectation value of the dual operator. We will review this argument below, and also derive the exact proportionality coefficient. Our result is in agreement with [81].

We proceed to regularise and then renormalise the theory. We regulate by integrating in the bulk from $\rho \geq \epsilon$,¹⁰

$$\begin{aligned}
S_{\text{M,reg}} &= \frac{1}{2} \int_{\rho \geq \epsilon} d^{d+1}x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) \\
&= - \int_{\rho=\epsilon} d^d x \sqrt{g(x, \epsilon)} \epsilon^{-\Delta+d/2} \left[\frac{1}{2} (d - \Delta) \phi^2(x, \epsilon) + \epsilon \phi(x, \epsilon) \partial_\epsilon \phi(x, \epsilon) \right] \quad (4.72) \\
&= \int d^d x \sqrt{g_{(0)}} [\epsilon^{-\Delta+d/2} a_{(0)}^{\text{M}} + \epsilon^{-\Delta+d/2+1} a_{(2)}^{\text{M}} + \dots + \epsilon a_{(2\Delta-d+2)}^{\text{M}} \\
&\quad - \log \epsilon a_{(2\Delta-d)}] + \mathcal{O}(\epsilon^0).
\end{aligned}$$

Clearly, with $\Delta - d/2$ a positive integer there is a finite number of divergent terms. The logarithmic divergence appears exactly when $\Delta = d/2 + k$, $k = 0, 1, \dots$, in agreement with the analysis in [96], and is directly related to the logarithmic term in (4.71). The first few of the power law divergences read

$$a_{(0)}^{\text{M}} = -\frac{1}{2}(d - \Delta)\phi_{(0)}^2, \quad a_{(2)}^{\text{M}} = -\frac{1}{4}\text{Tr } g_{(2)} \phi_{(0)}^2 + (d - \Delta + 1) \phi_{(0)} \phi_{(2)}. \quad (4.73)$$

Given a field of specific dimension it is straightforward to compute all divergent terms.

We now proceed to obtain the renormalised action by adding counter-terms to cancel the infinities,

$$\begin{aligned}
S_{\text{M,ren}} &= \lim_{\epsilon \rightarrow 0} [S_{\text{M,reg}} - \int d^d x \sqrt{g_{(0)}} [\epsilon^{-\Delta+d/2} a_{(0)}^{\text{M}} + \epsilon^{-\Delta+d/2+1} a_{(2)}^{\text{M}} + \dots + \epsilon a_{(2\Delta-d+2)}^{\text{M}} \\
&\quad - \log \epsilon a_{(2\Delta-d)}]. \quad (4.74)
\end{aligned}$$

Exactly as in the case of pure gravity, and since the regulated theory lives at $\rho = \epsilon$, one needs to rewrite the counter-terms in terms of the field living at $\rho = \epsilon$, i.e. the induced metric $\gamma_{ij}(x, \epsilon)$ and the field $\Phi(x, \epsilon)$, or equivalently $g_{ij}(x, \epsilon)$ and $\phi(x, \epsilon)$. This is straightforward but somewhat tedious: one needs to invert the relation between ϕ and $\phi_{(0)}$ and between g_{ij} and $g_{(0)ij}$ to sufficiently high order. This then allows to express all $\phi_{(n)}$, and therefore all $a_{(n)}^{\text{M}}$, in terms of $\phi(x, \epsilon)$ and $g_{ij}(x, \epsilon)$ (the $\phi_{(n)}$'s are determined in terms of $\phi_{(0)}$ and $g_{(0)}$ by solving (4.70) iteratively). Explicitly, the first two orders read

$$\begin{aligned}
S_{\text{M,ren}} &= \lim_{\epsilon \rightarrow 0} \left[\frac{1}{2} \int_{\rho \geq \epsilon} d^{d+1}x \sqrt{G} (G^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + m^2 \Phi^2) \right. \\
&\quad + \int_{\rho=\epsilon} \sqrt{\gamma} \left[\frac{(d - \Delta)}{2} \Phi^2(x, \epsilon) + \frac{1}{2(2\Delta - d - 2)} (\Phi(x, \epsilon) \square_\gamma \Phi(x, \epsilon) \right. \\
&\quad \left. \left. + \frac{d - \Delta}{2(d - 1)} R[\gamma] \Phi^2(x, \epsilon) + \dots \right) \right]. \quad (4.75)
\end{aligned}$$

¹⁰This regularisation for scalar fields in a fixed AdS background was considered in [90, 46]. In these papers the divergences were computed in momentum space, but no counter-terms were added to cancel them. Addition of boundary counter-terms to cancel infinities for scalar fields was considered in [28], and more recently in [81].

The addition of the first counter-term was discussed in [81]. The action (4.75) with only the counter-terms written explicitly is finite for fields of $\Delta < d/2 + 2$. As remarked above, it is straightforward to obtain all counter-terms needed in order to make the action finite for any field of any mass. These counter-terms contain also logarithmic subtractions that lead to the conformal anomalies discussed in [96]. For instance, if $\Delta = \frac{1}{2}d + 1$, the coefficient $[2(2\Delta - d - 2)]^{-1}$ in (4.75) is replaced by $-\frac{1}{4} \log \epsilon$. An alternative way to derive the counter-terms is to demand that the expectation value $\langle O \rangle$ is finite. This holds in the case of pure gravity too, i.e. the counter-terms can also be derived by requiring finiteness of $\langle T_{\mu\nu} \rangle$ [11].

The expectation value of the dual operator is given by

$$\langle O(x) \rangle = -\frac{1}{\sqrt{\det g_{(0)}}} \frac{\delta S_{\text{M,ren}}}{\delta \phi_{(0)}} = -\lim_{\epsilon \rightarrow 0} \frac{1}{\sqrt{\det g(x, \epsilon)}} \frac{\delta S_{\text{M,ren}}}{\delta \phi(x, \epsilon)}. \quad (4.76)$$

Exactly as in the case of pure gravity, the expectation value receives a contribution both from the regulated part and from the counter-terms. We obtain,

$$\langle O(x) \rangle = (2\Delta - d) \phi_{(2\Delta-d)} + F(\phi_{(n)}, \psi_{(2\Delta-d)}, g_{(m)}), \quad n < 2\Delta - d, \quad (4.77)$$

where we used that $\phi_{(2\Delta-d)}$ is linear in $\phi_{(0)}$ (notice that the action (4.67) does not include interactions). $F(\phi_{(n)}, \psi_{(2\Delta-d)}, g_{(m)})$ is a local function of $\phi_{(n)}$ with $n < 2\Delta - d$, $\psi_{(2\Delta-d)}$ and $g_{(m)}$. These terms are related to contact terms in correlation functions of O with itself and with the stress-energy tensor. Its exact form is straightforward but somewhat tedious to obtain (just use (4.75) and (4.76)).

As we have promised, we have shown that the coefficient $\phi_{(2\Delta-d)}$ is related with the expectation value of the dual CFT operator. In the case that the background geometry is the standard Euclidean AdS one can readily obtain $\phi_{(2\Delta-d)}$ from the unique solution of the scalar field equation with given Dirichlet boundary conditions. One finds that $\phi_{(2\Delta-d)}$ is proportional to (an integral involving) $\phi_{(0)}$. Therefore, $\phi_{(2\Delta-d)}$ carries information about the 2-point function. The factor $(\Delta - d/2)$ is crucial in order for the 2-point function to be normalised correctly [46]. We refer to [81] for a detailed discussion of this point.

We finish this section by calculating the conformal anomaly associated with the scalar fields and in the case the background is (locally) standard AdS (i.e. $g_{(n)} = 0$, for $0 < n < d$). Equation (4.70) simplifies and can be easily solved. One gets

$$\begin{aligned} \phi_{(2n)} &= \frac{1}{2n(2\Delta - d - 2n)} \square_0 \phi_{(2n-2)}, \\ \psi_{(2\Delta-d)} &= -\frac{1}{2(2\Delta - d)} \square_0 \phi_{(2\Delta-d-2)} = -\frac{1}{2^{2k} \Gamma(k) \Gamma(k+1)} (\square_0)^k \phi_{(0)}, \end{aligned} \quad (4.78)$$

where $k = \Delta - \frac{d}{2}$ and \square_0 is the Laplacian of $g_{(0)}$. The regularised action written in terms of the fields at $\rho = \epsilon$ contains the following explicit logarithmic divergence:

$$S_{\text{M,reg}} = -\int_{\rho=\epsilon} d^d x \sqrt{\gamma} [\log \epsilon (\Delta - \frac{d}{2}) \phi(x, \epsilon) \psi_{(2\Delta-d)}(x, \epsilon) + \dots], \quad (4.79)$$

where the dots indicate power law divergent and finite terms, $\psi_{(2\Delta-d)}(x, \epsilon)$ is given by (4.78) with $g_{(0)}$ replaced by γ and $\phi_{(0)}$ by $\phi(x, \epsilon)$. Using the same argument as in [70] we obtain the matter conformal anomaly,

$$\mathcal{A}_M = \frac{1}{2} \left(\frac{1}{2^{2k-2} (\Gamma(k))^2} \right) \phi_{(0)} (\Box_0)^k \phi_{(0)}. \quad (4.80)$$

This agrees exactly with the anomaly calculated in [96] (compare with formulae (10), (37) in [96]).

4.5.3 Scalars coupled to gravity

In the previous section we ignored the back-reaction of the scalars to the bulk geometry. The purpose of this section is to discuss this issue. The action is now the sum of (4.7) and (4.67),

$$S = S_{\text{gr}} + S_M. \quad (4.81)$$

The gravitational field equation in the presence of matter reads

$$R_{\mu\nu} - \frac{1}{2}(R + 2\Lambda)G_{\mu\nu} = -8\pi G_N T_{\mu\nu}. \quad (4.82)$$

In the co-ordinate system (4.9) and with the scalar field having the expansion in (4.71), these equations read

$$\begin{aligned} \rho [2g''_{ij} - 2(g'g^{-1}g')_{ij} + \text{Tr}(g^{-1}g')g'_{ij}] + R_{ij}(g) - (d-2)g'_{ij} - \text{Tr}(g^{-1}g')g_{ij} &= \quad (4.83) \\ &= -8\pi G_N \rho^{d-\Delta-1} \left[\frac{(\Delta-d)\Delta}{d-1} \phi^2 g_{ij} + \rho \partial_i \phi \partial_j \phi \right], \\ \nabla_i \text{Tr}(g^{-1}g') - \nabla^j g'_{ij} &= -16\pi G_N \rho^{d-\Delta-1} \left[\frac{d-\Delta}{2} \phi \partial_i \phi + \rho \partial_\rho \phi \partial_i \phi \right], \\ \text{Tr}(g^{-1}g'') - \frac{1}{2} \text{Tr}(g^{-1}g'g^{-1}g') &= -16\pi G_N \rho^{d-\Delta-2} \left[\frac{d(\Delta-d)(\Delta-d+1)}{4(d-1)} \phi^2 \right. \\ &\quad \left. + (d-\Delta) \rho \phi \partial_\rho \phi + \rho^2 (\partial_\rho \phi)^2 \right], \end{aligned}$$

If $\Delta > d$, the right-hand side diverges near the boundary whereas the left-hand side is finite. Operators with dimension $\Delta > d$ are irrelevant operators. Correlation functions of these operators have a very complicated singularity structure at coincident points. As remarked in [125], one can avoid such problems by considering the sources to be infinitesimal and to have disjoint support, so that these operators are never at coincident points. Requiring that the equations in (4.83) are satisfied to leading order in ρ yields

$$\phi_{(0)}^2 = 0, \quad (4.84)$$

which is indeed the prescription advocated in [125].

If $\Delta \leq d$, which means that we deal with marginal or relevant operators, one can perturbatively calculate the back-reaction of the scalars to the bulk metric. At which order the leading back-reaction appears depends on the mass of the field. For fields that correspond to operators of dimension $\Delta = d - k$ the leading back-reaction appears at order ρ^k , except when $k = 0$ (marginal operators), where the leading back-reaction is at order ρ .

Let us see how conformal anomalies arise in this context. The logarithmic divergences are coming from the regulated on-shell value of the bulk integral in (4.81). The latter reads

$$\begin{aligned} S_{\text{reg}}(\text{bulk}) &= \int_{\rho \geq \epsilon} d\rho d^d x \sqrt{G} \left[\frac{d}{8\pi G_N} - \frac{m^2}{d-1} \Phi^2 \right] \\ &= \int_{\rho \geq \epsilon} d\rho d^d x \frac{1}{\rho} \sqrt{g(x, \rho)} \left[\frac{d\rho^{-d/2}}{16\pi G_N} - \frac{m^2 \rho^{-k}}{2(d-1)} \phi^2(x, \rho) \right], \end{aligned} \quad (4.85)$$

where $k = \Delta - d/2$. We see that gravitational conformal anomalies are expected when d is even and matter conformal anomalies when k is a positive integer, as it should.

In the presence of sources the expectation value of the boundary stress-energy tensor is not conserved but rather it satisfies a Ward identity that relates its covariant divergence to the expectation value of the operators that couple to the sources. To see this consider the generating functional

$$Z_{\text{CFT}}[g_{(0)}, \phi_{(0)}] = \langle \exp \int d^d x \sqrt{g_{(0)}} \left[\frac{1}{2} g_{(0)}^{ij} T_{ij} - \phi_{(0)} O \right] \rangle. \quad (4.86)$$

Invariance under infinitesimal diffeomorphisms,

$$\delta g_{(0)ij} = \nabla_i \xi_j + \nabla_j \xi_i, \quad (4.87)$$

yields the Ward identity,

$$\nabla^j \langle T_{ij} \rangle = \langle O \rangle \partial_i \phi_{(0)}. \quad (4.88)$$

As we have remarked before, $\langle T_{ij} \rangle$ has a dual meaning [11], both as the expectation value of the dual stress-energy tensor and as the quasi-local stress-energy tensor of Brown and York. The Ward identity (4.88) has a natural explanation from the latter point in view as well. According to [25] the quasi-local stress-energy tensor is not conserved in the presence of matter but it satisfies

$$\nabla^j \langle T_{ij} \rangle = -\tau_{i\rho}, \quad (4.89)$$

where $\tau_{i\rho}$ expresses the flow of matter energy-momentum through the boundary. Evidently, (4.88) is of the form (4.89).

Solving the coupled system of equations (4.83) and (4.70) is straightforward but somewhat tedious. The details differ from case to case. For illustrative purposes we present a sample calculation: we consider the case of two-dimensional massless scalar field ($d = \Delta = 2, k = 1$).

The equations to be solved are (4.70) and (4.83) with $d = \Delta = 2$ and the expansion of the metric and the scalar field are given by (4.9) and (4.71) (again with $d = \Delta = 2$), respectively. Equation (4.70) determines $\psi_{(2)}$,

$$\psi_{(2)} = -\frac{1}{4}\square_0\phi_{(0)}. \quad (4.90)$$

Equations (4.83) determine $h_{(2)}$, the trace of the $g_{(2)}$ and provide a relation between the divergence of $g_{(2)}$ and $\phi_{(2)}$,

$$\begin{aligned} h_{(2)} &= -4\pi G_N \left(\partial_i \phi_{(0)} \partial_j \phi_{(0)} - \frac{1}{2} g_{(0)ij} (\partial \phi_{(0)})^2 \right), \\ \text{Tr } g_{(2)} &= \frac{1}{2} R + 4\pi G_N (\partial \phi_{(0)})^2, \\ \nabla^i g_{(2)ij} &= \partial_i \text{Tr } g_{(2)} + 16\pi G_N \phi_{(2)} \partial_i \phi_{(0)}. \end{aligned} \quad (4.91)$$

Notice that $g_{(2)}$ and $\phi_{(2)}$ are still undetermined and are related to the expectation values of the dual operators (4.27) and (4.77), respectively. Notice that $h_{(2)}$ is equal to the stress-energy tensor of a massless two-dimensional scalar.

Going back to (4.85), we see that the second term drops out (since $m^2 = 0$) and one can use the result already obtained in the gravitational sector,

$$\begin{aligned} \mathcal{A} &= \frac{1}{16\pi G_N} (-2a_{(2)}) = \frac{1}{16\pi G_N} (-2\text{Tr } g_{(2)}) \\ &= -\frac{1}{16\pi G_N} R + \frac{1}{2} \phi_{(0)} \square_0 \phi_{(0)} - \frac{1}{2} \nabla_i (\phi_{(0)} \nabla^i \phi_{(0)}), \end{aligned} \quad (4.92)$$

which is the correct conformal anomaly [70, 96] (the last term can be removed by adding a covariant counter-term).

The renormalised boundary stress tensor reads

$$\langle T_{ij}(x) \rangle = \frac{1}{8\pi G_N} (g_{(2)ij} + h_{(2)ij} - g_{(0)ij} \text{Tr } g_{(2)}) (x). \quad (4.93)$$

Its trace gives correctly the conformal anomaly (4.92). On the other hand, taking the covariant derivative of (4.93) we get

$$\begin{aligned} \nabla^j \langle T_{ij} \rangle &= \langle O(x) \rangle \partial_i \phi_0(x) \quad , \\ \langle O(x) \rangle &= 2(\phi_2(x) + \psi_2(x)). \end{aligned} \quad (4.94)$$

in agreement with equations (4.88) and (4.77).

4.5.4 Pointlike particles

The method developed in the previous subsections is quite generic and can be applied to other matter fields. Although we have not worked out all the details, in this section we give a further example for illustrative purposes: we consider pointlike particles. This is in our opinion a very important example for our understanding of holography in the

AdS/CFT correspondence, and we hope to report the full details elsewhere. Indeed, one can do interesting gedanken experiments with point particles and black holes in AdS [98, 108, 85] to test the causality and locality properties of the boundary theory.

So we couple the Einstein-Hilbert action to the action for a pointlike particle. One then needs to solve Einstein's equations coupled to the geodesic equation and the constraint

$$G_{\mu\nu}(z)\dot{z}^\mu\dot{z}^\nu = -\varepsilon \quad (4.95)$$

($\varepsilon = 1$ for massive particles and $\varepsilon = 0$ for massless particles). In the massive case, we get the following stress-energy tensor:

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{|G(x)|}} \int dt \delta^{(d+1)}(x - z(t)) \dot{z}^\mu \dot{z}^\nu. \quad (4.96)$$

In the massless case, the stress-energy tensor is given by (2.5). We have also analysed the tachyonic case, but we will not present the results here.

We are interested in computing the back-reaction effects of the particle on the metric near the boundary. This will allow us to compute the expectation value of the stress-energy tensor of the dual theory [35], which will depend in a crucial manner on the boundary conditions on the position and the speed of the particle. Therefore we are interested in the asymptotic behaviour of the stress-energy tensor as $r \rightarrow 0$. This is given by the part of the trajectory satisfying $r(t) \rightarrow 0$. Hence, the problem of finding the asymptotics of the stress-energy tensor translates itself into finding the region of the trajectory γ near the boundary. For the massless particle and the tachyon it is known that they can travel from the boundary to the bulk and viceversa, so we expect that there are values of t corresponding to $r = 0$. However, the particle with positive mass squared never reaches the boundary, and so we expect it not to contribute to the stress-energy tensor at $r = 0$. As we will see, this turns out to be true also for Einstein spaces with arbitrary boundary metric.

The strategy will be the following. To identify the region of t for which $r(t) \rightarrow 0$, we solve the geodesic equation perturbatively in r and find the solutions $r(t)$ and $z^i(t)$ perturbatively in some function of t . If there are such solutions, the perturbative expansion makes sense; if there are not, the geodesic equation cannot be solved perturbatively near the boundary.

The massless particle

To lowest order, the geodesic equations for massless particles are solved by

$$\begin{aligned} r(t) &= \frac{1}{c(t-d)} \\ z^i(t) &= z_0^i + r(t)v^i, \end{aligned} \quad (4.97)$$

where v^i is now a timelike vector, $g_{ij}v^iv^j = -1$, defined in general by $v^i(r) \equiv \frac{dz^i}{dr}$. In this case, the stress-energy tensor can be cast in the form

$$T_{\mu\nu} = \frac{pc\ell^2}{\sqrt{g}} \left(\frac{r}{\ell}\right)^{d+3} v_\mu v_\nu \delta^{(d)}(x - z(r)), \quad (4.98)$$

where $v^\mu(r)$ is defined by $v^\mu(r) = (1, v^i(r))$. It is null in the space-time metric and satisfies

$$\begin{aligned}\partial_\mu v^\mu &= 0 \\ v^\mu \partial_\mu \delta^{(d)}(x - z(r)) &= 0.\end{aligned}\tag{4.99}$$

All components of the above stress-energy tensor are proportional to r^{d-1} in leading order in r as $r \rightarrow 0$. Therefore, it will contribute to $g_{(d)}$ but not to $h_{(d)}$, just as in the tachyonic case. It is now also straightforward to compute the dual stress-energy tensor. This will have an interesting behaviour [72]: to start with, unless one chooses very special boundary conditions, the effective Hamiltonian will be time-dependent due to the covariance of our formulae in the boundary co-ordinates. Notice that to get agreement with the results in [72], where the stress-energy is centred on the light-cone, one may need to first perform a co-ordinate transformation. As mentioned in the previous sections, such a co-ordinate transformation changes the value of the stress-energy tensor if it induces a boundary Weyl rescaling.

The massive particle

The bulk trajectory of a particle with positive mass squared is given by

$$r(t) = \frac{r_0}{|\cos(t/\ell + c)|},\tag{4.100}$$

and, like in the tachyonic case, r_0 and c are to be determined by the boundary conditions only. In this case, however, we see that $r(t)$ can never be zero unless $r_0 = 0$, in which case the particle stays forever at the boundary and never reaches the bulk. Therefore, a perturbative solution of the geodesic equation in powers of r does not make sense in this case, as the world-line of the particle actually never reaches the boundary. Therefore, one can only hope to solve the geodesic equation for simple exact solutions of the vacuum Einstein equations. For example, it is an elementary exercise to solve for the case of a flat boundary, the trajectory being given by (4.100). In that case, one finds an expression for the stress-energy tensor analogous to that for the tachyon, but now involving a step function $\theta(r - r_0)$, hence with support only on the region $r > r_0$.

In this case, the particle contributes only a finite piece to the action.

4.6 Conclusions

Most of the discussions in the literature on the AdS/CFT correspondence are concerned with obtaining conformal field theory correlation functions using supergravity. Here we started investigating the converse question: how can one obtain information about the bulk theory from CFT correlation functions? How does one decode the hologram?

Answering these questions in all generality, but within the context of the AdS/CFT duality, entails developing a precise dictionary between bulk and boundary physics. A prescription for relating bulk/boundary observables is already available [64, 125], and one would expect that it would allow us to reconstruct the bulk space-time from the

boundary CFT. The prescription of [64, 125], however, relates infinite quantities. One of the main results presented here is the systematic development of a renormalised version of this prescription. Equipped with it, and with no other assumption (except that the CFT has an AdS dual), we then proceeded to reconstruct the bulk space-time metric and bulk scalar fields to the first non-trivial order.

Our approach to the problem is to start from the boundary and try to build iteratively bulk solutions. Within this approach, the pattern we find is the following:

- Sources in the CFT determine an asymptotic expansion of the corresponding bulk field near the boundary to high enough order so that *all infrared divergences* of the bulk on-shell action can be computed. This then allows to obtain a renormalised on-shell action by adding boundary counter-terms to cancel the infrared divergences.
- Bulk solutions can be extended one order further by using the 1-point function of the corresponding dual CFT operator.

In the case the bulk field is the metric, our results show that a conformal structure at infinity is not in general sufficient in order to obtain a bulk metric. The first additional information one needs is the expectation value of the boundary stress-energy tensor.

As a by-product, we have obtained ready-to-use formulae for the Brown-York quasi-local stress-energy tensor for arbitrary solution of Einstein's equations with negative cosmological constant up to six dimensions. The six-dimensional result is particularly interesting because, via AdS/CFT, it provides new information about the still mysterious $(2, 0)$ theory. Furthermore, we have obtained the conformal transformation properties of the stress-energy tensors. These transformation rules incorporate the trace anomaly and provide a generalisation to $d > 2$ of the well-known Schwartzian derivative contribution in the conformal transformation rule of the stress-energy tensor in $d = 2$.

Our discussion extends straightforwardly to the case of different matter. We expect that in all cases obstructions in extending the solution to the deep interior region will be resolved by additional CFT data. An interesting case to study in this framework is point particles. Reconstructing the trajectory of the bulk point particle out of CFT data will present a model of how holography works with time dependent processes. Furthermore, following [72], one could study the interplay between causality and holography. Another extension is to study renormalisation group flows using the present formalism. This amounts to extending the discussion in section 4.2 by adding a potential for the scalars. Another application of our results is in the context of Randall-Sundrum (RS) scenarios [99]. Incorporating such a scenario in string theory, in the case the bulk space is AdS, may yield a connection with the AdS/CFT duality [123, 126]. As advocated in [126], one may view the RS scenario as $4d$ gravity coupled to a cut-off CFT. The regulated theory in our discussion provides a dual description of a cut-off CFT. In this context, the counter-terms are re-interpreted as providing the action for the bulk modes localised on the brane [102, 62, 55]. We see, for instance, that the counter-terms in (4.75) can be re-interpreted as an action for a bulk scalar mode localised on the brane (see, e.g., [31]). This is the subject of study in the next chapter.