Chapter 2

Indefinite ϑ-functions

2.1 Introduction

The classical theta series associated to a positive definite quadratic form $Q : \mathbf{R}^r \longrightarrow \mathbf{R}$ and $B : \mathbf{R}^r \times \mathbf{R}^r \longrightarrow \mathbf{R}$, the associated bilinear form $B(x, y) = Q(x+y) - Q(x) - Q(y)$, is the series

$$\Theta(z; \tau) := \sum_{n \in \mathbf{Z}^r} e^{2\pi i Q(n)\tau + 2\pi i B(n,z)}. \tag{2.1}$$

These theta series have well-known transformation properties. In particular $\Theta(0; \tau)$ is a modular form of weight $r/2$.

In [11] Göttsche and Zagier define a theta function for the case when the type of Q is $(r-1, 1)$. The definition of these functions is almost the same as in (2.1), only here the sum doesn't run over $\mathbf{Z}^r$, but some appropriate subset. However, in general, these functions do not have nice modular transformation properties.

In this chapter we give a modified definition. We find elliptic and modular transformation properties for these functions. The theta functions we define depend not only on Q, but also on two vectors $c_1, c_2 \in \mathbf{R}^r$ with $Q(c_i) \leq 0$, $i = 1, 2$. The case $Q(c_1) = Q(c_2) = 0$ gives the same functions as in [11].

There is a connection between the indefinite ϑ-functions from this chapter and certain ϑ-functions considered by Siegel (see [25]). However, I will not give this connection here.

2.2 Definition of ϑ

Let A be a symmetric $r \times r$-matrix with integer coefficients, which is non-degenerate. We consider the quadratic form $Q : \mathbf{C}^r \longrightarrow \mathbf{C}$, $Q(x) = \frac{1}{2}\langle x, Ax\rangle$ and the associated bilinear form $B(x, y) = \langle x, Ay\rangle = Q(x + y) - Q(x) - Q(y)$.

The *type* of Q is the pair $(r - s, s)$, where s is the largest dimension of a linear subspace of $\mathbf{R}^r$ on which Q is negative definite. The *signature* of Q is the number

$r-2s$.

From now on we assume that $s=1$, i.e., that Q has type $(r-1,1)$. Then the set of vectors $c \in \mathbf{R}^r$ with $Q(c)<0$ has two components. If $B(c_1,c_2)<0$ then c_1 and c_2 belong to the same component, while if $B(c_1,c_2)>0$ then c_1 and c_2 belong to opposite components. Let C_Q be one of the two components. If c_0 is a vector in that component, then C_Q is given by:

$$C_Q := \{c \in \mathbf{R}^r \mid Q(c)<0,\ B(c,c_0)<0\}.$$

We further set

$$S_Q := \{c \in \mathbf{Z}^r \mid c \text{ primitive},\ Q(c)=0,\ B(c,c_0)<0\}.$$

(c primitive means that the greatest common divisor of the components of c is 1). The $(r-1)$-dimensional *hyperbolic space* $C_Q/\mathbf{R}_+$ is the natural domain of definition of automorphic forms with respect to $O_A^+(\mathbf{Z})$ (see section 2.4 for the definition), and S_Q is a set of representatives for the corresponding set of *cusps*

$$\{c \in \mathbf{Q}^r \mid Q(c)=0,\ B(c,c_0)<0\}/\mathbf{Q}_+.$$

Note that S_Q is empty in some cases, for example if $A=\left(\begin{smallmatrix}1 & 0\\ 0 & -3\end{smallmatrix}\right)$. Further we put $\overline{C}_Q := C_Q \cup S_Q$. This is a generalisation of the usual construction $\overline{\mathcal{H}} = \mathcal{H} \cup \mathbb{P}^1(\mathbf{Q})$, which is the special cone $\overline{C}_Q = C_Q \cup S_Q$ for the quadratic form $Q(a,b,c)=\frac{1}{2}(b^2-4ac)$.

For $c \in \overline{C}_Q$ put

$$R(c) := \begin{cases} \mathbf{R}^r & \text{if } c \in C_Q \\ \{a \in \mathbf{R}^r \mid B(c,a) \notin \mathbf{Z}\} & \text{if } c \in S_Q \end{cases}$$

and

$$D(c) := \{(z,\tau) \in \mathbf{C}^r \times \mathcal{H} \mid \operatorname{Im}(z)/\operatorname{Im}(\tau) \in R(c)\}.$$

Definition 2.1 Let $c_1, c_2 \in \overline{C}_Q$. We define the *theta function* of Q with *characteristics* $a \in R(c_1) \cap R(c_2)$ and $b \in \mathbf{R}^r$, with respect to (c_1,c_2) by

$$\vartheta_{a,b}(\tau) = \vartheta_{a,b}^{c_1,c_2}(\tau) := \sum_{\nu \in a+\mathbf{Z}^r} \rho(\nu;\tau) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)},$$

where $\rho(\nu;\tau)$ is defined by

$$\rho(\nu;\tau) = \rho_A^{c_1,c_2}(\nu;\tau) := \rho^{c_1}(\nu;\tau) - \rho^{c_2}(\nu;\tau),$$

with

$$\rho^c(\nu;\tau) = \begin{cases} E\left(\frac{B(c,\nu)}{\sqrt{-Q(c)}} y^{1/2}\right) & \text{if } c \in C_Q, \\ \operatorname{sgn}(B(c,\nu)) & \text{if } c \in S_Q, \end{cases}$$

with $y = \mathrm{Im}(\tau)$ and E as in Definition 1.6.

For $(z,\tau) \in D(c_1) \cap D(c_2)$, we define the theta function of Q with respect to (c_1, c_2) by

$$\begin{aligned}\vartheta(z;\tau) = \vartheta_A^{c_1,c_2}(z;\tau) &:= e^{-2\pi i Q(a)\tau - 2\pi i B(a,b)}\vartheta_{a,b}(\tau) \\ &= \sum_{n\in\mathbf{Z}^r} \rho(n+a;\tau) e^{2\pi i Q(n)\tau + 2\pi i B(n,z)},\end{aligned}$$

with $a, b \in \mathbf{R}^r$ defined by $z = a\tau + b$, so $a = \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}$, $b = \frac{\mathrm{Im}(\overline{z}\tau)}{\mathrm{Im}(\tau)}$.

Remark 2.2 The definition doesn't change if we replace c_i by λc_i, with $\lambda \in \mathbf{R}_+$. Hence we could replace the condition $Q(c_i) < 0$ by $Q(c_i) = -1$. This would simplify the definition of ρ.

Remark 2.3 In some special cases $\vartheta_{a,b}$ is holomorphic: if $c_1, c_2 \in S_Q$ and, as we will see in section 2.5, also for some special values of c_1, c_2, a and b. In general however, the functions ϑ and $\vartheta_{a,b}$ are not holomorphic.

Because Q is indefinite, $e^{2\pi i Q(n)\tau}$ isn't bounded. Therefore it's not immediately clear that the series defining $\vartheta(z;\tau)$ converges absolutely. However, using an estimate for the growth of ρ we shall find:

Proposition 2.4 *The series defining $\vartheta(z;\tau)$ converges absolutely.*

For the proof of this proposition, we need two lemmas

Lemma 2.5 *Let $c \in C_Q$. The quadratic form $Q_c : \mathbf{R}^r \longrightarrow \mathbf{R}$, $Q_c(\nu) := Q(\nu) - \frac{B(c,\nu)^2}{2Q(c)}$ is positive definite, and we have*

$$Q_c(\nu) \geq \lambda_{c,c_0} Q_{c_0}(\nu) \qquad \forall \nu \in \mathbf{R}^r,$$

with

$$\lambda_{c,c_0} = \frac{B(c,c_0)^2 - 2Q(c)Q(c_0) - |B(c,c_0)|\sqrt{B(c,c_0)^2 - 4Q(c)Q(c_0)}}{2Q(c)Q(c_0)} > 0.$$

Proof: If $\nu \in \mathbf{R}^r$ is linearly independent of c, the quadratic form Q has type $(1,1)$ on span$\{c,\nu\}$; hence the matrix

$$\begin{pmatrix} 2Q(c) & B(c,\nu) \\ B(c,\nu) & 2Q(\nu) \end{pmatrix}$$

has determinant < 0, so $4Q(\nu)Q(c) - B(c,\nu)^2 < 0$. Rewriting gives

$$Q(\nu) - \frac{B(c,\nu)^2}{2Q(c)} > -\frac{B(c,\nu)^2}{4Q(c)} \geq 0.$$

If $\nu = \lambda c$, with $\lambda \neq 0$, we get $Q(\nu) - \frac{B(c,\nu)^2}{2Q(c)} = -Q(c)\lambda^2 > 0$, which proves that Q_c is positive definite.

For the second part we consider the restriction of Q_c to the ellipsoid $S = \{\nu \in \mathbf{R}^r | Q_{c_0}(\nu) = 1\}$. Since S is compact, $Q_c|_S$ assumes its absolute minimum at some point ν_0. We compute that minimum with the method of Lagrange multipliers: There is a real number λ, such that

$$\nabla Q_c(\nu_0) = \lambda \nabla Q_{c_0}(\nu_0),$$

or equivalently

$$A\left(\nu_0 - \frac{B(\nu_0,c)}{Q(c)}c\right) = \lambda A\left(\nu_0 - \frac{B(\nu_0,c_0)}{Q(c_0)}c_0\right). \tag{2.2}$$

Taking the inner product with c on both sides of (2.2) we find

$$-B(\nu_0,c) = \lambda\left(B(\nu_0,c) - \frac{B(\nu_0,c_0)B(c,c_0)}{Q(c_0)}\right). \tag{2.3}$$

Taking the inner product with c_0 on both sides of (2.2) we find

$$B(\nu_0,c_0) - \frac{B(\nu_0,c)B(c,c_0)}{Q(c)} = -\lambda B(\nu_0,c_0). \tag{2.4}$$

Combining (2.3) and (2.4) we find

$$Q(c)Q(c_0)(\lambda+1)^2 = \lambda B(c,c_0)^2 \tag{2.5}$$

or

$$B(\nu_0,c) = B(\nu_0,c_0) = 0.$$

If $B(\nu_0,c) = B(\nu_0,c_0) = 0$, then (2.2) reduces to $A\nu_0 = \lambda A\nu_0$, from which we find $\lambda = 1$.

The roots of (2.5) are

$$\lambda^{\pm} = \frac{B(c,c_0)^2 - 2Q(c)Q(c_0) \pm |B(c,c_0)|\sqrt{B(c,c_0)^2 - 4Q(c)Q(c_0)}}{2Q(c)Q(c_0)}.$$

Taking the inner product with ν_0 on both sides of (2.2) and dividing by 2, we find

$$Q_c(\nu_0) = \lambda Q_{c_0}(\nu_0) = \lambda.$$

Hence the absolute minimum of $Q_c|_S$ is the minimum of $\{1, \lambda^-, \lambda^+\}$ which is $\lambda^- = \lambda_{c,c_0}$. So we have

$$Q_c(\nu) \geq \lambda_{c,c_0} \qquad \forall \nu \in S.$$

Let $\nu \in \mathbf{R}^r$, $\nu \neq 0$, then $\frac{\nu}{\sqrt{Q_{c_0}(\nu)}} \in S$, so

$$\lambda_{c,c_0} \leq Q_c\left(\frac{\nu}{\sqrt{Q_{c_0}(\nu)}}\right) = \frac{Q_c(\nu)}{Q_{c_0}(\nu)}.$$

Multiplying both sides by $Q_{c_0}(\nu)$ we get the desired result. □

Lemma 2.6 *Let $c_1, c_2 \in C_Q$ be linearly independent. The quadratic form $Q^+ : \mathbf{R}^r \longrightarrow \mathbf{R}$, $Q^+(\nu) := Q(\nu) + \frac{B(c_1,c_2)}{4Q(c_1)Q(c_2)-B(c_1,c_2)^2}B(c_1,\nu)B(c_2,\nu)$ is positive definite.*

Proof: If $\nu \in \mathbf{R}^r$ is not a linear combination of c_1 and c_2, the quadratic form Q has type $(2,1)$ on $\operatorname{span}\{c_1, c_2, \nu\}$; so the matrix

$$\begin{pmatrix} 2Q(c_1) & B(c_1,c_2) & B(c_1,\nu) \\ B(c_1,c_2) & 2Q(c_2) & B(c_2,\nu) \\ B(c_1,\nu) & B(c_2,\nu) & 2Q(\nu) \end{pmatrix} \tag{2.6}$$

has determinant < 0. Rewriting gives

$$Q^+(\nu) > \frac{Q(c_2)B(c_1,\nu)^2 + Q(c_1)B(c_2,\nu)^2}{4Q(c_1)Q(c_2) - B(c_1,c_2)^2} \geq 0.$$

If $\nu \in \mathbf{R}^r$ is a linear combination of c_1 and c_2 the determinant of the matrix in (2.6) is zero. Hence

$$Q^+(\nu) = \frac{Q(c_2)B(c_1,\nu)^2 + Q(c_1)B(c_2,\nu)^2}{4Q(c_1)Q(c_2) - B(c_1,c_2)^2}.$$

So if $Q^+(\nu) = 0$, we have $B(c_1,\nu) = B(c_2,\nu) = 0$, which implies $\nu = 0$. Thus if $\nu \neq 0$ then $Q^+(\nu)$ is strictly positive, so Q^+ is positive definite.

Proof of Proposition 2.4: If $c_1, c_2 \in C_Q$, we write $\rho(\nu;\tau)$, using Lemma 1.7, as the sum of the three expressions

$$-\operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right), \tag{2.7}$$

$$\operatorname{sgn}\Big(B(c_2,\nu)\Big)\beta\left(-\frac{B(c_2,\nu)^2}{Q(c_2)}y\right) \tag{2.8}$$

and

$$\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big), \tag{2.9}$$

with β as in Lemma 1.7. If $c_1 \in C_Q$ and $c_2 \in S_Q$ we get only the sum of the first and the last expression. If $c_1 \in S_Q$ and $c_2 \in C_Q$ we get the sum of the last two expressions. If $c_1, c_2 \in S_Q$ we have only the last expression. Hence the proof is reduced to showing that the series

$$\sum_{\nu \in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} \tag{2.10}$$

converges absolutely for all c with $Q(c) < 0$, and that the series

$$\sum_{\nu \in a+\mathbf{Z}^r} \Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\} e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} \tag{2.11}$$

converges absolutely for all $c_1, c_2 \in \overline{C}_Q$.

We will first show that the series (2.10) converges absolutely for all c with $Q(c) < 0$: We can easily see that $0 \le \beta(x) \le e^{-\pi x}$ for all $x \in \mathbf{R}_{\ge 0}$; hence if $Q(c) < 0$

$$\begin{aligned} &\left|\operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right)e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right| \\ &\le e^{\pi\frac{B(c,\nu)^2}{Q(c)}y}\left|e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)}\right| \\ &= e^{-2\pi\left(Q(\nu)-\frac{B(c,\nu)^2}{2Q(c)}\right)y}. \end{aligned} \tag{2.12}$$

Using Lemma 2.5, we see that the series

$$\sum_{\nu\in a+\mathbf{Z}^r} e^{-2\pi\left(Q(\nu)-\frac{B(c,\nu)^2}{2Q(c)}\right)y}$$

converges, and so the series (2.10) converges absolutely if $Q(c) < 0$.

We will now show that the series (2.11) converges absolutely for all $c_1, c_2 \in \overline{C}_Q$: If c_1 and c_2 are linearly dependent, we have $\rho^{c_1,c_2} = 0$. Hence we can assume that they are linearly independent.

Case 1: $c_1, c_2 \in C_Q$.

If we have $B(c_1,\nu)B(c_2,\nu) > 0$, then $\operatorname{sgn}(B(c_1,\nu)) - \operatorname{sgn}(B(c_2,\nu)) = 0$. If we have $B(c_1,\nu)B(c_2,\nu) \le 0$, then (note that $4Q(c_1)Q(c_2) - B(c_1,c_2)^2 < 0$, as we saw before)

$$Q(\nu) \ge Q^+(\nu),$$

with Q^+ as in Lemma 2.6. Hence we find

$$\begin{aligned} &\left|\left\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\right\}e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)}\right| \\ &= \left|\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\right| e^{-2\pi Q(\nu)y} \\ &\le 2e^{-2\pi Q^+(\nu)y}. \end{aligned} \tag{2.13}$$

Using Lemma 2.6, we see that the series

$$\sum_{\nu\in a+\mathbf{Z}^r} e^{-2\pi Q^+(\nu)y}$$

converges, and so the series (2.11) converges absolutely.

Case 2: $c_1 \in C_Q$ and $c_2 \in S_Q$.

We can assume that $c_1 \in C_Q \cap \mathbf{Z}^r$, since otherwise we pick any $c_1' \in C_Q \cap \mathbf{Z}^r$, write

$$\begin{aligned} &\sum_{\nu\in a+\mathbf{Z}^r}\left\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\right\} e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)} \\ &\quad= \sum_{\nu\in a+\mathbf{Z}^r}\left\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_1',\nu)\Big)\right\} e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)} \\ &\quad+ \sum_{\nu\in a+\mathbf{Z}^r}\left\{\operatorname{sgn}\Big(B(c_1',\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\right\} e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)}, \end{aligned}$$

and use that

$$\sum_{\nu\in a+\mathbf{Z}^r}\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_1',\nu)\Big)\Big\}\;e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}$$

converges absolutely.

We write $\nu=\mu+nc_2$ with $\mu\in a+\mathbf{Z}^r$ and $n\in\mathbf{Z}$, such that $\frac{B(c_1,\mu)}{B(c_1,c_2)}\in[0,1)$ (we see $n=\Big[\frac{B(c_1,\nu)}{B(c_1,c_2)}\Big]$). Then $\operatorname{sgn}(B(c_2,\nu))=\operatorname{sgn}(B(c_2,\mu))$ (use $B(c_2,c_2)=0$) and $\operatorname{sgn}(B(c_1,\nu))=-\operatorname{sgn}(n+\frac{B(c_1,\mu)}{B(c_1,c_2)})$. Hence

$$\begin{aligned}\sum_{\nu\in a+\mathbf{Z}^r}&\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\;e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}\\ &=-\sum_{\substack{\mu\in a+\mathbf{Z}^r\\ \frac{B(c_1,\mu)}{B(c_1,c_2)}\in[0,1)}}\;\sum_{n\in\mathbf{Z}}\Big\{\operatorname{sgn}\Big(B(c_2,\mu)\Big)+\operatorname{sgn}\Big(n+\frac{B(c_1,\mu)}{B(c_1,c_2)}\Big)\Big\}\cdot\\ &\qquad\qquad\cdot e^{2\pi iQ(\mu)\tau+2\pi iB(c_2,\mu)n\tau+2\pi iB(\mu,b)+2\pi iB(c_2,b)n}\end{aligned}$$

Using

$$\frac{1}{1-x}=\begin{cases}\sum_{n=0}^{\infty}x^n & \text{if } |x|<1\\ -\sum_{n=-\infty}^{-1}x^n & \text{if } |x|>1,\end{cases}$$

we see

$$\begin{aligned}\sum_{n\in\mathbf{Z}}&\Big\{\operatorname{sgn}\Big(B(c_2,\mu)\Big)+\operatorname{sgn}\Big(n+\frac{B(c_1,\mu)}{B(c_1,c_2)}\Big)\Big\}\;e^{2\pi iB(c_2,\mu)n\tau+2\pi iB(c_2,b)n}\\ &=\frac{2}{1-e^{2\pi iB(c_2,\mu)\tau+2\pi iB(c_2,b)}}-\delta(B(c_1,\mu)).\end{aligned}$$

Here we used that

$$B(c_2,\mu)\geq B(c_2,\tilde\mu)>0, \tag{2.14}$$

for all $\mu\in a+\mathbf{Z}^r$, and for some $\tilde\mu\in a+\mathbf{Z}^r$. This is guaranteed by the fact that $(z,\tau)\in D(c_2)$.

Since $c_1,c_2\in\mathbf{Z}^r$ we have

$$\Big\{\mu\in a+\mathbf{Z}^r\;\Big|\;\frac{B(c_1,\mu)}{B(c_1,c_2)}\in[0,1)\Big\}=\bigcup_{\mu_0\in P_0}\Big(\mu_0+\langle c_1\rangle_{\mathbf{Z}}^{\perp}\Big),$$

for a suitable finite set P_0, with $\langle c_1\rangle_{\mathbf{Z}}^{\perp}:=\{\xi\in\mathbf{Z}^r\mid B(c_1,\xi)=0\}$. So

$$\begin{aligned}\sum_{\nu\in a+\mathbf{Z}^r}&\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\;e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}\\ &=-\sum_{\mu_0\in P_0}\;\sum_{\xi\in\langle c_1\rangle^{\perp}}\Big\{\frac{2}{1-e^{2\pi iB(c_2,\xi+\mu_0)\tau+2\pi iB(c_2,b)}}-\delta(B(c_1,\mu_0))\Big\}\cdot\\ &\qquad\qquad e^{2\pi iQ(\xi+\mu_0)\tau+2\pi iB(\xi+\mu_0,b)}.\end{aligned}$$

This series converges absolutely, since Q is positive definite on $\langle c_1\rangle_{\mathbf{Z}}^{\perp}$, and the term

$$\frac{2}{1-e^{2\pi i B(c_2,\xi+\mu_0)\tau+2\pi i B(c_2,b)}}-\delta(B(c_1,\mu_0))$$

is bounded (use (2.14)).
Case 3: $c_1\in S_Q$ and $c_2\in C_Q$.
Since $\vartheta^{c_1,c_2}=-\vartheta^{c_2,c_1}$, this follows directly from the previous case.
Case 4: $c_1,c_2\in S_Q$.
Since $\vartheta^{c_1,c_2}=\vartheta^{c_1,c_3}+\vartheta^{c_3,c_2}$, for arbitrary $c_3\in C_Q$, this follows directly from case 2 and 3. □

2.3 Properties of the ϑ-functions

The theta functions in Definition 2.1 have some nice elliptic and modular transformation properties, similar to those of the theta functions associated to positive definite quadratic forms.

Proposition 2.7 *The function ϑ satisfies:*

(1) For $c_1,c_2,c_3\in\overline{C}_Q$ and $(z,\tau)\in D(c_1)\cap D(c_2)\cap D(c_3)$ we have the cocycle conditions $\vartheta^{c_1,c_2}+\vartheta^{c_2,c_1}=0$ and $\vartheta^{c_1,c_2}+\vartheta^{c_2,c_3}+\vartheta^{c_3,c_1}=0$.

(2) $\vartheta(z+\lambda\tau+\mu;\tau)=e^{-2\pi i Q(\lambda)\tau-2\pi i B(z,\lambda)}\vartheta(z;\tau)$ for all $\lambda\in\mathbf{Z}^r$ and $\mu\in A^{-1}\mathbf{Z}^r$.

(3) $\vartheta(-z;\tau)=-\vartheta(z;\tau)$.

(4) The function $(c_1,c_2)\mapsto\vartheta^{c_1,c_2}$ is continuous on $C_Q\times C_Q$.

(5) Let $c_1,c_3\in C_Q$, $c_2\in S_Q$ and $(z,\tau)\in D(c_2)$. Set $c(t)=c_2+tc_3$. Then $c(t)\in C_Q$ for all $t\in(0,\infty)$ and $\lim_{t\downarrow 0}\vartheta^{c_1,c(t)}(z;\tau)=\vartheta^{c_1,c_2}(z;\tau)$.

(6) $\vartheta(z;\tau+1)=\vartheta(z+\frac{1}{2}A^{-1}A^*;\tau)$ with $A^*=(A_{11}\dots A_{rr})^T\in\mathbf{Z}^r$, the vector of diagonal elements of A. In particular, $\vartheta(z;\tau+2)=\vartheta(z;\tau)$ and $\vartheta(z;\tau+1)=\vartheta(z;\tau)$ if the matrix A is even.

(7) Let $D'(c):=\left\{(z,\tau)\in D(c)\mid\left(\frac{z}{\tau},-\frac{1}{\tau}\right)\in D(c)\right\}=\left\{(a\tau+b,\tau)\mid\tau\in\mathcal{H},\ a,b\in\mathbf{R}^r,\ B(c,a)\notin\mathbf{Z},\ B(c,b)\notin\mathbf{Z}\right\}$. If $(z,\tau)\in D'(c_1)\cap D'(c_2)$ then

$$\vartheta\Big(\frac{z}{\tau};-\frac{1}{\tau}\Big)=\frac{i}{\sqrt{-\det A}}(-i\tau)^{r/2}\sum_{p\in A^{-1}\mathbf{Z}^r/\mathbf{Z}^r}e^{2\pi i Q(z+p\tau)/\tau}\vartheta(z+p\tau;\tau).$$

Proof: (1) follows from the corresponding relations for ρ^{c_1,c_2}.
(2) The identity $\vartheta(z+\mu;\tau)=\vartheta(z;\tau)$ for $\mu\in A^{-1}\mathbf{Z}^r$ is easy, and we find $\vartheta(z+\lambda\tau;\tau)=e^{-2\pi i Q(\lambda)\tau-2\pi i B(z,\lambda)}\vartheta(z;\tau)$ for $\lambda\in\mathbf{Z}^r$ when we replace n by $n+\lambda$ in the definition.

For (3), replace n by $-n$ in the definition and use that E and sgn are odd functions.
(4) We show that $c_1 \mapsto \vartheta^{c_1,c_2}$ is continuous on C_Q. The result then follows from (1).

Using the decomposition of ρ as the sum of (2.7), (2.8) and (2.9) we see that it's sufficient to prove that

$$c \mapsto \sum_{\nu \in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}, \tag{2.15}$$

and

$$c_1 \mapsto \sum_{\nu \in a+\mathbf{Z}^r} \Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\; e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} \tag{2.16}$$

are continuous on C_Q.

Using Lemma 2.5 and (2.12) we see

$$\begin{aligned}\left|\operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right| &\le e^{-2\pi\left(Q(\nu)-\frac{B(c,\nu)^2}{2Q(c)}\right)y}\\ &\le e^{-2\pi\lambda_{c,c_0}Q_{c_0}(\nu)y}\end{aligned}$$

Since $c \mapsto \lambda_{c,c_0}$ is continuous and $\lambda_{c,c_0} > 0$ for all $c \in C_Q$, we can find an neighbourhood $\mathcal{N}_c$ of c such that $\lambda_{c,c_0} \ge \epsilon > 0$ for all $c \in \mathcal{N}_c$. Hence on $\mathcal{N}_c$ we find

$$\left|\operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right| \le e^{-2\pi\epsilon Q_{c_0}(\nu)y}.$$

The series

$$\sum_{\nu \in a+\mathbf{Z}^r} e^{-2\pi\epsilon Q_{c_0}(\nu)y}$$

converges, and so the series in (2.15) converges uniformly for c in $\mathcal{N}_c$. Hence the function in (2.15) is continuous on $\mathcal{N}_c$. Since this holds for all $c \in C_Q$, the function in (2.15) is continuous on C_Q.

In (2.13) we have seen that

$$\left|\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\; e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right| \le 2e^{-2\pi Q^+(\nu)y}.$$

The function Q^+ restricted to the sphere $S = \{\nu \in \mathbf{R}^r |\; \|\nu\| = 1\}$ assumes its absolute minimum $\lambda(c_1) > 0$. Hence

$$Q^+(\nu) \ge \lambda(c_1) \qquad \forall \nu \in S,$$

and so

$$Q^+(\nu) \ge \lambda(c_1)\|\nu\|^2 \qquad \forall \nu \in \mathbf{R}^r.$$

Since $c_1 \mapsto \lambda(c_1)$ is continuous and $\lambda(c_1) > 0$ for all $c_1 \in C_Q$, we can find an neighbourhood $\mathcal{N}_{c_1}$ of c_1 such that $\lambda(c_1) \geq \epsilon > 0$ for all $c_1 \in \mathcal{N}_{c_1}$. Hence on $\mathcal{N}_{c_1}$ we find

$$\left|\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\ e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right| \leq 2e^{-2\pi\epsilon\|\nu\|^2 y}.$$

The series

$$\sum_{\nu\in a+\mathbf{Z}^r} e^{-2\pi\epsilon\|\nu\|^2 y}$$

converges, and so the series in (2.16) converges uniformly for c_1 in $\mathcal{N}_{c_1}$. Hence the function in (2.16) is continuous on $\mathcal{N}_{c_1}$. Since this holds for all $c_1 \in C_Q$, the function in (2.16) is continuous on C_Q.

(5) Note that $\vartheta^{c_1,c(t)} = \vartheta^{c_1,c_3} + \vartheta^{c_3,c(t)}$. We can therefore assume c_3 to be equal to c_1. We have $Q(c(t)) = Q(c_2 + tc_1) = tB(c_1,c_2) + t^2Q(c_1) < 0$ and $B(c_1,c(t)) = B(c_1,c_2) + 2tQ(c_1) < 0$ for all $t \in (0,\infty)$, since $B(c_1,c_2) < 0$ and $Q(c_1) < 0$. Hence $c(t) \in C_Q$ for all $t \in (0,\infty)$.

Using $\vartheta^{c_1,c(t)} = \vartheta^{c_1,c_2} + \vartheta^{c_2,c(t)}$ and the decomposition of ρ as the sum of (2.7), (2.8) and (2.9) we see that it's sufficient to prove that

$$\lim_{t\downarrow 0} \sum_{\nu\in a+\mathbf{Z}^r} \Big\{\operatorname{sgn}\Big(B(c_2,\nu)\Big) - \operatorname{sgn}\Big(B(c(t),\nu)\Big)\Big\}\ e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} = 0, \tag{2.17}$$

and

$$\lim_{t\downarrow 0} \sum_{\nu\in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c(t),\nu)\Big)\beta\left(-\frac{B(c(t),\nu)^2}{Q(c(t))}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} = 0. \tag{2.18}$$

It is easy to see that

$$\left|\operatorname{sgn}\Big(B(c_2,\nu)\Big) - \operatorname{sgn}\Big(B(c(t),\nu)\Big)\right| \leq \left|\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\right|$$

for all $\nu \in a+\mathbf{Z}^r$ and $t \in (0,\infty)$ (Both sides can take on the values 0,1 and 2. If the right hand side is 0, then $\operatorname{sgn}(B(c_1,\nu)) = \operatorname{sgn}(B(c_2,\nu))$, so $\operatorname{sgn}(B(c_2,\nu)) = \operatorname{sgn}(B(c(t),\nu))$. Hence the left hand side is also 0, and the equation holds. If the right hand side is 1, then either $B(c_1,\nu)$ or $B(c_2,\nu)$ is zero. If $B(c_2,\nu) = 0$ we get that the left hand side equals the right hand side. If $B(c_1,\nu) = 0$ the left hand side equals 0). Hence

$$\begin{aligned}&\left|\Big\{\operatorname{sgn}\Big(B(c_2,\nu)\Big) - \operatorname{sgn}\Big(B(c(t),\nu)\Big)\Big\}\ e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right|\\ &\qquad \leq \left|\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\ e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right|,\end{aligned}$$

for all $\nu \in a + \mathbf{Z}^r$ and $t \in (0,\infty)$. In the proof of Proposition 2.4 (Case 2) we have seen that (2.11) converges absolutely, i.e.

$$\sum_{\nu\in a+\mathbf{Z}^r} \left|\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\ e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}\right|$$

converges. Hence

$$\sum_{\nu\in a+\mathbf{Z}^r}\Big\{\operatorname{sgn}\Big(B(c_2,\nu)\Big)-\operatorname{sgn}\Big(B(c(t),\nu)\Big)\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}$$

converges uniformly for $t\in(0,\infty)$. Using this we find

$$\begin{aligned}\lim_{t\downarrow 0}&\sum_{\nu\in a+\mathbf{Z}^r}\Big\{\operatorname{sgn}\Big(B(c_2,\nu)\Big)-\operatorname{sgn}\Big(B(c(t),\nu)\Big)\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}\\ &=\sum_{\nu\in a+\mathbf{Z}^r}\lim_{t\downarrow 0}\Big\{\operatorname{sgn}\Big(B(c_2,\nu)\Big)-\operatorname{sgn}\Big(B(c(t),\nu)\Big)\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}=0.\end{aligned}$$

This proves (2.17).

We will now prove (2.18): Using (2.12), we see that

$$\left|\operatorname{sgn}\Big(B(c(t),\nu)\Big)\beta\left(-\frac{B(c(t),\nu)^2}{Q(c(t))}y\right)e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}\right|\leq e^{-2\pi\left(Q(\nu)-\frac{B(c(t),\nu)^2}{2Q(c(t))}\right)y}.$$

We write $a+\mathbf{Z}^r$ as the union of P_1, P_2 and P_3, with

$$\begin{aligned}P_1&:=\{\nu\in a+\mathbf{Z}^r\mid\operatorname{sgn}(B(c_2,\nu))=-\operatorname{sgn}(B(c_1,\nu))\}\\ P_2&:=\{\nu\in a+\mathbf{Z}^r\mid B(c_1,\nu)(B(c_1,c_2)B(c_1,\nu)-2Q(c_1)B(c_2,\nu))\geq 0\}\\ P_3&:=\{\nu\in a+\mathbf{Z}^r\mid\operatorname{sgn}(B(c_2,\nu))=-\operatorname{sgn}(B(c_1,c_2)B(c_1,\nu)-2Q(c_1)B(c_2,\nu))\}\end{aligned}$$

Note that $B(c_2,\nu)\neq 0$ for all $\nu\in a+\mathbf{Z}^r$, which is guaranteed by the fact that $(z,\tau)\in D(c_2)$.

On P_1 we use

$$e^{-2\pi\left(Q(\nu)-\frac{B(c(t),\nu)^2}{2Q(c(t))}\right)y}\leq e^{-2\pi Q(\nu)y},$$

for all $t\in(0,\infty)$. We have seen in the proof of Proposition 2.4 (Case 2) that the series in (2.11) converges absolutely. Hence the series

$$\sum_{\nu\in P_1}e^{-2\pi Q(\nu)y}$$

converges.

On P_2 we have

$$\frac{B(c(t),\nu)^2}{2Q(c(t))}\leq\frac{B(c_1,\nu)^2}{2Q(c_1)}$$

for all $t\in(0,\infty)$, which we get from

$$B(c_2,\nu)^2+\left(2B(c_2,\nu)B(c_1,\nu)-\frac{B(c_1,c_2)B(c_1,\nu)^2}{Q(c_1)}\right)t\geq 0$$

for all $t \in (0,\infty)$. Hence we find

$$e^{-2\pi\left(Q(\nu)-\frac{B(c(t),\nu)^2}{2Q(c(t))}\right)y} \leq e^{-2\pi\left(Q(\nu)-\frac{B(c_1,\nu)^2}{2Q(c_1)}\right)y},$$

for all $t \in (0,\infty)$. Using Lemma 2.5 we see that the series

$$\sum_{\nu\in P_2} e^{-2\pi\left(Q(\nu)-\frac{B(c_1,\nu)^2}{2Q(c_1)}\right)y}$$

converges.

On P_3 we use

$$\frac{B(c(t),\nu)^2}{2Q(c(t))} \leq \frac{2B(c_2,\nu)}{B(c_1,c_2)^2}\Big(B(c_1,c_2)B(c_1,\nu)-Q(c_1)B(c_2,\nu)\Big), \tag{2.19}$$

for all $t \in (0,\infty)$, which we get from the inequality

$$\left(B(c_2,\nu)+\left(-B(c_1,\nu)+\frac{2Q(c_1)}{B(c_1,c_2)}B(c_2,\nu)\right)t\right)^2 \geq 0.$$

Note that (2.19) holds also on P_1 and P_2, but we use it only on P_3. Using it we find

$$e^{-2\pi\left(Q(\nu)-\frac{B(c(t),\nu)^2}{2Q(c(t))}\right)y} \leq e^{-2\pi\widetilde{Q}(\nu)},$$

for all $t \in (0,\infty)$, with

$$\widetilde{Q}(\nu) := Q(\nu) - \frac{2B(c_2,\nu)}{B(c_1,c_2)^2}\Big(B(c_1,c_2)B(c_1,\nu)-Q(c_1)B(c_2,\nu)\Big). \tag{2.20}$$

Write $\nu = \nu_{c_1}c_1 + \nu_{c_2}c_2 + \nu^\perp$, with $\nu^\perp$ such that $B(c_1,\nu^\perp)=B(c_2,\nu^\perp)=0$. We see

$$\begin{aligned}B(c_1,\nu) &= 2Q(c_1)\nu_{c_1} + B(c_1,c_2)\nu_{c_2}\\ B(c_2,\nu) &= B(c_1,c_2)\nu_{c_1},\end{aligned}$$

so

$$\begin{aligned}\nu_{c_1} &= \frac{1}{B(c_1,c_2)}B(c_2,\nu)\\ \nu_{c_2} &= \frac{1}{B(c_1,c_2)}B(c_1,\nu) - \frac{2Q(c_1)}{B(c_1,c_2)^2}B(c_2,\nu).\end{aligned}$$

Hence

$$\begin{aligned}Q(\nu) &= Q(\nu^\perp) + Q(c_1)\nu_{c_1}^2 + B(c_1,c_2)\nu_{c_1}\nu_{c_2}\\ &= Q(\nu^\perp) + \frac{B(c_2,\nu)}{B(c_1,c_2)^2}\Big(B(c_1,c_2)B(c_1,\nu)-Q(c_1)B(c_2,\nu)\Big)\end{aligned}$$

and

$$\widetilde{Q}(\nu) = Q(\nu^{\perp}) - \frac{B(c_2,\nu)}{B(c_1,c_2)^2}\Big(B(c_1,c_2)B(c_1,\nu) - Q(c_1)B(c_2,\nu)\Big).$$

The quadratic form $\widetilde{Q}$ has type $(r-1,1)$: Q has type $(1,1)$ on $\langle c_1, c_2\rangle_{\mathbf{R}}$ and type $(r-2,0)$ on $\langle c_1,c_2\rangle_{\mathbf{R}}^{\perp}$. On $\langle c_1,c_2\rangle_{\mathbf{R}}^{\perp}$ we have $\widetilde{Q} = Q$ and on $\langle c_1,c_2\rangle_{\mathbf{R}}$ we have $\widetilde{Q} = -Q$. Hence $\widetilde{Q}$ has type $(1,1)$ on $\langle c_1,c_2\rangle_{\mathbf{R}}$ and type $(r-2,0)$ on $\langle c_1,c_2\rangle_{\mathbf{R}}^{\perp}$.

Set $\tilde{c}_1 = \frac{B(c_1,c_2)}{2Q(c_1)}c_1 - c_2$ and $\tilde{c}_2 = -c_2$ then

$$\begin{aligned}
\widetilde{Q}(\tilde{c}_1) &= \widetilde{Q}\left(\frac{B(c_1,c_2)}{2Q(c_1)}c_1 - c_2\right) = -Q\left(\frac{B(c_1,c_2)}{2Q(c_1)}c_1 - c_2\right)\\
&= -\frac{B(c_1,c_2)^2}{4Q(c_1)} + \frac{B(c_1,c_2)^2}{2Q(c_1)} = \frac{B(c_1,c_2)^2}{4Q(c_1)} < 0\\
\widetilde{Q}(\tilde{c}_2) &= \widetilde{Q}(-c_2) = -Q(c_2) = 0\\
\widetilde{B}(\tilde{c}_1,\tilde{c}_2) &= -\widetilde{Q}(\tilde{c}_1 - \tilde{c}_2) + \widetilde{Q}(\tilde{c}_1) + \widetilde{Q}(\tilde{c}_2)\\
&= Q\left(\frac{B(c_1,c_2)}{2Q(c_1)}c_1\right) + \frac{B(c_1,c_2)^2}{4Q(c_1)} = \frac{B(c_1,c_2)^2}{2Q(c_1)} < 0.
\end{aligned}$$

If we choose $C_{\widetilde{Q}}$ such that $\tilde{c}_1 \in C_{\widetilde{Q}}$ then we see that $\tilde{c}_2 \in S_{\widetilde{Q}}$.

Using (2.20) we see

$$\begin{aligned}
\widetilde{B}(x,y) &= \widetilde{Q}(x+y) - \widetilde{Q}(x) - \widetilde{Q}(y)\\
&= B(x,y) - \frac{2}{B(c_1,c_2)}\Big(B(c_2,x)B(c_1,y) + B(c_1,x)B(c_2,y)\Big)\\
&\qquad + 4\frac{Q(c_1)}{B(c_1,c_2)^2}B(c_2,x)B(c_2,y),
\end{aligned}$$

so

$$\begin{aligned}
\widetilde{B}(c_1,\nu) &= -B(c_1,\nu)\\
\widetilde{B}(c_2,\nu) &= -B(c_2,\nu).
\end{aligned}$$

Since $\tilde{c}_1$ and $\tilde{c}_2$ are linear combinations of c_1 and c_2, we have

$$\begin{aligned}
\widetilde{B}(\tilde{c}_1,\nu) &= -B(\tilde{c}_1,\nu)\\
\widetilde{B}(\tilde{c}_2,\nu) &= -B(\tilde{c}_2,\nu)
\end{aligned}$$

for all $\nu \in \mathbf{R}^r$.

We rewrite the set P_3:

$$\begin{aligned}
P_3 &= \{\nu \in a+\mathbf{Z}^r \mid \operatorname{sgn}(B(c_2,\nu)) = -\operatorname{sgn}(B(c_1,c_2)B(c_1,\nu) - 2Q(c_1)B(c_2,\nu))\}\\
&= \{\nu \in a+\mathbf{Z}^r \mid \operatorname{sgn}(B(-c_2,\nu)) = -\operatorname{sgn}\left(\frac{B(c_1,c_2)}{2Q(c_1)}B(c_1,\nu) - B(c_2,\nu)\right)\}\\
&= \{\nu \in a+\mathbf{Z}^r \mid \operatorname{sgn}(B(\tilde{c}_2,\nu)) = -\operatorname{sgn}(B(\tilde{c}_1,\nu))\}\\
&= \{\nu \in a+\mathbf{Z}^r \mid \operatorname{sgn}(\widetilde{B}(\tilde{c}_2,\nu)) = -\operatorname{sgn}(\widetilde{B}(\tilde{c}_1,\nu))\}.
\end{aligned}$$

In the proof of Proposition 2.4 (Case 2, (2.11)), we have seen that

$$\sum_{\nu\in a+\mathbf{Z}^r} \left\{\operatorname{sgn}\Big(\widetilde{B}(\tilde{c}_1,\nu)\Big) - \operatorname{sgn}\Big(\widetilde{B}(\tilde{c}_2,\nu)\Big)\right\} e^{2\pi i\widetilde{Q}(\nu)\tau+2\pi i\widetilde{B}(\nu,b)}$$

converges absolutely, i.e. the series

$$\sum_{\nu\in P_3} e^{-2\pi\widetilde{Q}(\nu)y}$$

converges.

On all three sets P_1, P_2 and P_3, we have found a suitable majorant, independent of $t\in(0,\infty)$. Combining these results, we see that the series

$$\sum_{\nu\in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c(t),\nu)\Big)\beta\left(-\frac{B(c(t),\nu)^2}{Q(c(t))}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}$$

converges uniformly for $t\in(0,\infty)$. Hence

$$\begin{aligned}
&\lim_{t\downarrow 0}\sum_{\nu\in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c(t),\nu)\Big)\beta\left(-\frac{B(c(t),\nu)^2}{Q(c(t))}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}\\
&\quad= \sum_{\nu\in a+\mathbf{Z}^r} \lim_{t\downarrow 0}\operatorname{sgn}\Big(B(c(t),\nu)\Big)\beta\left(-\frac{B(c(t),\nu)^2}{Q(c(t))}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}.
\end{aligned}$$

We have

$$\lim_{t\downarrow 0} -\frac{B(c(t),\nu)^2}{Q(c(t))}y = \infty$$

and

$$\lim_{x\to\infty}\beta(x) = 0.$$

Hence we get equation (2.18).

(6) Since $\rho(a;\tau)$ depends only on $\operatorname{Im}(\tau)$, we have $\rho(a;\tau+1)=\rho(a;\tau)$. Hence

$$\vartheta(z;\tau+1) = \sum_{n\in\mathbf{Z}^r} \rho(n+a;\tau)e^{2\pi iQ(n)}e^{2\pi iQ(n)\tau+2\pi iB(n,z)}, \tag{2.21}$$

but since A has integer coefficients we find

$$e^{2\pi i Q(n)} = e^{\pi i \sum_{l=1}^{r} A_{ll} n_l^2} = e^{\pi i \sum_{l=1}^{r} A_{ll} n_l} = e^{2\pi i B(\frac{1}{2}A^{-1}A^*, n)}.$$

If we put this into (2.21) we get (6).

We first prove (7) for the case $c_1, c_2 \in C_Q$. We do this using the Poisson summation formula. The main point – and the reason for the definition of the function ρ – is that $a \mapsto \rho(a;\tau) e^{2\pi i Q(a)\tau}$ is more or less its own Fourier transform:

Lemma 2.8 *We have for all* $\alpha \in \mathbf{R}^r$ *and* $\tau \in \mathcal{H}$

$$\int_{\mathbf{R}^r} \rho(a;\tau)\ e^{2\pi i Q(a)\tau + 2\pi i B(a,\alpha)} da = \frac{1}{\sqrt{-\det A}} \frac{i}{(-i\tau)^{r/2}} \rho\Big(\alpha; -\frac{1}{\tau}\Big)\ e^{-2\pi i Q(\alpha)/\tau}.$$

Proof: The integral converges. This is analogous to the convergence of ϑ for case 1: We write $\rho(a;\tau)$ as the sum of the three expressions

$$-\operatorname{sgn}\Big(B(c_1,a)\Big)\ \beta\left(-\frac{B(c_1,a)^2}{Q(c_1)} y\right),$$
$$\operatorname{sgn}\Big(B(c_2,a)\Big)\ \beta\left(-\frac{B(c_2,a)^2}{Q(c_2)} y\right)$$

and

$$\operatorname{sgn}\Big(B(c_1,a)\Big) - \operatorname{sgn}\Big(B(c_2,a)\Big).$$

We have

$$\begin{aligned} &\left|\operatorname{sgn}\Big(B(c,a)\Big)\ \beta\left(-\frac{B(c,a)^2}{Q(c)} y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(a,b)}\right| \\ &\le e^{-2\pi\left(Q(a) - \frac{B(c,a)^2}{2Q(c)}\right) y}, \end{aligned}$$

(see (2.12)), with $a \mapsto Q(a) - \frac{B(c,a)^2}{2Q(c)}$ positive definite (see Lemma 2.5). We also have

$$\left|\Big\{\operatorname{sgn}\Big(B(c_1,a)\Big) - \operatorname{sgn}\Big(B(c_2,a)\Big)\Big\}\ e^{2\pi i Q(a)\tau + 2\pi i B(a,b)}\right| \le 2e^{-2\pi Q^+(a) y},$$

(see (2.13)), with Q^+ positive definite (see Lemma 2.6).

Using

$$\frac{\partial}{\partial \alpha_l} e^{2\pi i Q(a\tau+\alpha)/\tau} = \frac{1}{\tau} \frac{\partial}{\partial a_l} e^{2\pi i Q(a\tau+\alpha)/\tau}$$

we see that

$$\begin{aligned} &\frac{\partial}{\partial \alpha_l}\left\{ e^{2\pi i Q(\alpha)/\tau} \int_{\mathbf{R}^r} \rho(a;\tau)\ e^{2\pi i Q(a)\tau + 2\pi i B(a,\alpha)} da \right\} \\ &\quad = \frac{\partial}{\partial \alpha_l} \int_{\mathbf{R}^r} \rho(a;\tau)\ e^{2\pi i Q(a\tau+\alpha)/\tau} da = \int_{\mathbf{R}^r} \rho(a;\tau) \frac{\partial}{\partial \alpha_l} e^{2\pi i Q(a\tau+\alpha)/\tau} da \qquad (2.22) \\ &\quad = \int_{\mathbf{R}^r} \rho(a;\tau) \frac{1}{\tau} \frac{\partial}{\partial a_l} e^{2\pi i Q(a\tau+\alpha)/\tau} da = -\frac{1}{\tau} \int_{\mathbf{R}^r} \frac{\partial \rho}{\partial a_l}(a;\tau)\ e^{2\pi i Q(a\tau+\alpha)/\tau} da, \end{aligned}$$

where we have used partial integration in the last step. From the definition of ρ it follows that

$$\begin{aligned}&\frac{\partial\rho}{\partial a_l}(a;\tau)\\&\quad=\frac{(Ac_1)_l}{\sqrt{-Q(c_1)}}\;y^{1/2}\;E'\Big(\frac{B(c_1,a)}{\sqrt{-Q(c_1)}}\;y^{1/2}\Big)-\frac{(Ac_2)_l}{\sqrt{-Q(c_2)}}\;y^{1/2}\;E'\Big(\frac{B(c_2,a)}{\sqrt{-Q(c_2)}}\;y^{1/2}\Big).\end{aligned}\tag{2.23}$$

We have (we will use this result for $c=c_1$ and $c=c_2$)

$$\begin{aligned}&\int_{\mathbf{R}^r}E'\Big(\frac{B(c,a)}{\sqrt{-Q(c)}}y^{1/2}\Big)\;e^{2\pi iQ(a\tau+\alpha)/\tau}da\\&\qquad\qquad=2e^{2\pi iQ(\alpha)/\tau}\int_{\mathbf{R}^r}e^{\pi\frac{B(c,a)^2}{Q(c)}y}e^{2\pi iQ(a)\tau+2\pi iB(a,\alpha)}da.\end{aligned}$$

We substitute $a=\left(\begin{smallmatrix}c & C\end{smallmatrix}\right)\left(\begin{smallmatrix}a_c\\ a'\end{smallmatrix}\right)$, with $a_c\in\mathbf{R}$, $a'\in\mathbf{R}^{r-1}$ and C a $r\times(r-1)$-matrix whose columns form a basis for

$$\langle c\rangle_{\mathbf{R}}^{\perp}:=\{a\in\mathbf{R}^r\mid B(c,a)=0\}.$$

In that way we can split the integral over $\mathbf{R}^r$ in an integral over $\mathbf{R}$ and an integral over $\mathbf{R}^{r-1}$ (Note that $B(c,Ca')=0$, hence $Q(a)=Q(c)a_c^2+\frac{1}{2}\left\langle a',C^TACa'\right\rangle$ and $B(c,a)=2Q(c)a_c$):

$$\begin{aligned}&y^{1/2}\int_{\mathbf{R}^r}E'\Big(\frac{B(c,a)}{\sqrt{-Q(c)}}y^{1/2}\Big)\;e^{2\pi iQ(a\tau+\alpha)/\tau}da\\&\quad=2y^{1/2}e^{2\pi iQ(\alpha)/\tau}\int_{\mathbf{R}\times\mathbf{R}^{r-1}}e^{4\pi Q(c)a_c^2y+2\pi iQ(c)a_c^2\tau+\pi i\left\langle a',C^TACa'\right\rangle\tau}\cdot\\&\qquad\qquad\qquad e^{4\pi iQ(c)a_c\alpha_c+2\pi i\left\langle a',C^TAC\alpha'\right\rangle}\left|\det\begin{pmatrix}c & C\end{pmatrix}\right|da'da_c\\&\quad=2\left|\det\begin{pmatrix}c & C\end{pmatrix}\right|y^{1/2}e^{2\pi iQ(\alpha)/\tau}\cdot\\&\qquad\int_{\mathbf{R}}e^{2\pi iQ(c)a_c^2\overline{\tau}+4\pi iQ(c)a_c\alpha_c}da_c\cdot\int_{\mathbf{R}^{r-1}}e^{\pi i\left\langle a',C^TACa'\right\rangle\tau+2\pi i\left\langle a',C^TAC\alpha'\right\rangle}da',\end{aligned}$$

with $\alpha=\left(\begin{smallmatrix}c & C\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha_c\\ \alpha'\end{smallmatrix}\right)$.

If $\tau\in\mathcal{H}$ and M is a positive definite symmetric $n\times n$-matrix, we have the well known result

$$\int_{\mathbf{R}^n}e^{\pi i\langle a,Ma\rangle\tau+2\pi i\langle a,M\alpha\rangle}da=\frac{1}{(-i\tau)^{n/2}}\frac{1}{\sqrt{\det M}}e^{-\pi i\langle\alpha,M\alpha\rangle/\tau}.$$

(By a change of basis in $\mathbf{R}^n$ one can reduce to the case when M is diagonal).

Q is positive definite on $\langle c\rangle_{\mathbf{R}}^{\perp}$, so C^TAC is positive definite. Hence we find using the result twice:

$$\begin{aligned}
&y^{1/2}\int_{\mathbf{R}^r} E'\Big(\frac{B(c,a)}{\sqrt{-Q(c)}}y^{1/2}\Big)\ e^{2\pi iQ(a\tau+\alpha)/\tau}da\\
&\quad = 2\left|\det\begin{pmatrix}c & C\end{pmatrix}\right| y^{1/2}e^{2\pi iQ(\alpha)/\tau}\frac{1}{\sqrt{-2iQ(c)\overline{\tau}}}\ e^{-2\pi iQ(c)\alpha_c^2/\overline{\tau}}.\\
&\qquad\qquad\qquad\qquad\qquad\frac{1}{(-i\tau)^{(r-1)/2}}\frac{1}{\sqrt{\det C^TAC}}\ e^{-\pi i\left\langle\alpha',C^TAC\alpha'\right\rangle/\tau}\\
&\quad = \frac{2y^{1/2}}{\sqrt{i\overline{\tau}}}\frac{1}{(-i\tau)^{(r-1)/2}}\frac{\left|\det\begin{pmatrix}c & C\end{pmatrix}\right|}{\sqrt{-2Q(c)\det C^TAC}}\ e^{2\pi iQ(c)\alpha_c^2/\tau-2\pi iQ(c)\alpha_c^2/\overline{\tau}}\\
&\quad = \frac{2\sqrt{y'}}{(-i\tau)^{r/2-1}}\frac{\left|\det\begin{pmatrix}c & C\end{pmatrix}\right|}{\sqrt{-2Q(c)\det C^TAC}}\ e^{\pi\frac{B(c,\alpha)^2}{Q(c)}y'}\\
&\quad = \frac{\sqrt{y'}}{(-i\tau)^{r/2-1}}\frac{1}{\sqrt{-\det A}}\ E'\Big(\frac{B(c,\alpha)}{\sqrt{-Q(c)}}\sqrt{y'}\Big),
\end{aligned}$$

with $y'=\operatorname{Im}\left(-\frac{1}{\tau}\right)$. In the last step we have used

$$\left(\det\left(\begin{smallmatrix}c & C\end{smallmatrix}\right)\right)^2\det A = 2Q(c)\det C^TAC,$$

which follows from

$$\begin{pmatrix}c & C\end{pmatrix}^T A\begin{pmatrix}c & C\end{pmatrix} = \begin{pmatrix}2Q(c) & 0\\ 0 & C^TAC\end{pmatrix}$$

by taking the determinant.

We see

$$\begin{aligned}
&-\frac{1}{\tau}\int_{\mathbf{R}^r}\frac{(Ac)_l}{\sqrt{-Q(c)}}\ y^{1/2}E'\Big(\frac{B(c,a)}{\sqrt{-Q(c_1)}}y^{1/2}\Big)\ e^{2\pi iQ(a\tau+\alpha)/\tau}da\\
&\qquad = -\frac{1}{\tau}\frac{(Ac)_l}{\sqrt{-Q(c)}}\frac{\sqrt{y'}}{(-i\tau)^{r/2-1}}\frac{1}{\sqrt{-\det A}}\ E'\Big(\frac{B(c,\alpha)}{\sqrt{-Q(c)}}\sqrt{y'}\Big)\\
&\qquad = \frac{\partial}{\partial\alpha_l}\frac{1}{\sqrt{-\det A}}\frac{i}{(-i\tau)^{r/2}}\ E\Big(\frac{B(c,\alpha)}{\sqrt{-Q(c)}}\sqrt{y'}\Big).
\end{aligned}$$

Combining this with (2.22) and (2.23) we find

$$\begin{aligned}
&\frac{\partial}{\partial\alpha_l}\left\{e^{2\pi iQ(\alpha)/\tau}\int_{\mathbf{R}^r}\rho(a;\tau)e^{2\pi iQ(a)\tau+2\pi iB(a,\alpha)}da\right\}\\
&\qquad = \frac{\partial}{\partial\alpha_l}\frac{1}{\sqrt{-\det A}}\frac{i}{(-i\tau)^{r/2}}\ \rho\Big(\alpha;-\frac{1}{\tau}\Big).
\end{aligned}$$

So

$$e^{2\pi i Q(\alpha)/\tau}\int_{\mathbf{R}^r}\rho(a;\tau)\ e^{2\pi i Q(a)\tau+2\pi i B(a,\alpha)}da-\frac{1}{\sqrt{-\det A}}\frac{i}{(-i\tau)^{r/2}}\ \rho\Big(\alpha;-\frac{1}{\tau}\Big)$$

is constant as a function of α. Since both terms are odd as a function of α, that constant is zero. This proves the lemma. □

Proof of (7): *Case 1:* $c_1, c_2 \in C_Q$.

Using the Poisson summation formula

$$\sum_{\nu\in\mathbf{Z}^r} f(\nu)=\sum_{\nu\in A^{-1}\mathbf{Z}^r}\tilde{f}(\nu),$$

with $\tilde{f}(\nu)=\int_{\mathbf{R}^r} f(a)\ e^{2\pi i B(\nu,a)}da$, and Lemma 2.8, we find that $\vartheta_{a,b}$ satisfies

$$\vartheta_{a,b}\Big(-\frac{1}{\tau}\Big)=\frac{i}{\sqrt{-\det A}}\ (-i\tau)^{r/2}e^{2\pi i B(a,b)}\sum_{p\in A^{-1}\mathbf{Z}^r \bmod \mathbf{Z}^r}\vartheta_{b+p,-a}(\tau). \tag{2.24}$$

If we put

$$\vartheta_{a,b}(\tau)=e^{2\pi i Q(a)\tau+2\pi i B(a,b)}\ \vartheta(a\tau+b;\tau)$$

into (2.24) (on the left replace (a,b,τ) by $(a,b,-1/\tau)$, on the right by $(b+p,-a,\tau)$) and multiply both sides by $e^{2\pi i Q(a)/\tau-2\pi i B(a,b)}$, then we find

$$\vartheta\Big(\frac{b\tau-a}{\tau};-\frac{1}{\tau}\Big)=\frac{i}{\sqrt{-\det A}}\ (-i\tau)^{r/2}\sum_{p\in A^{-1}\mathbf{Z}^r/\mathbf{Z}^r} e^{2\pi i Q(b\tau-a+p\tau)/\tau}\vartheta(b\tau-a+p\tau;\tau),$$

which is the desired result for $z=b\tau-a$.

Case 2: $c_1\in C_Q$ and $c_2\in S_Q$.

We use (5): We have proven the identity for $\vartheta^{c_1,c(t)}$; if we take $\lim_{t\downarrow 0}$ on both sides we get the desired result.

The other two cases follow using the cocycle conditions given in (1). □

Corollary 2.9 *The function $\vartheta_{a,b}$ has the following elliptic and modular transformation properties:*

(1) $\vartheta_{a+\lambda,b}=\vartheta_{a,b}$ for all $\lambda\in\mathbf{Z}^r$.

(2) $\vartheta_{a,b+\mu}=e^{2\pi i B(a,\mu)}\ \vartheta_{a,b}$ for all $\mu\in A^{-1}\mathbf{Z}^r$.

(3) $\vartheta_{-a,-b}=-\vartheta_{a,b}$.

(4) $\vartheta_{a,b}(\tau+1)=e^{-2\pi i Q(a)-\pi i B(A^{-1}A^*,a)}\ \vartheta_{a,a+b+\frac{1}{2}A^{-1}A^*}(\tau)$ with A^* the vector of diagonal elements of A.

(5) If $a,b\in R(c_1)\cap R(c_2)$ then

$$\vartheta_{a,b}\Big(-\frac{1}{\tau}\Big)=\frac{i}{\sqrt{-\det A}}\ (-i\tau)^{r/2}e^{2\pi i B(a,b)}\sum_{p\in A^{-1}\mathbf{Z}^r \bmod \mathbf{Z}^r}\vartheta_{b+p,-a}(\tau).$$

2.4 Transformation properties of ϑ with respect to $O_A^+(\mathbf{Z})$

We consider the group

$$O_A(\mathbf{R}) := \{C \in \mathrm{GL}_r(\mathbf{R}) \mid C^t A C = A\}.$$

If $C \in O_A(\mathbf{R})$ and $c \in C_Q \subset \mathbf{R}^r$ then $Q(Cc) = Q(c)$, so $C \cdot C_Q$ is either C_Q or $-C_Q$. We consider only matrices C that leave C_Q invariant, i.e. $B(Cc, c) < 0$, for all $c \in C_Q$. Set

$$O_A^+(\mathbf{R}) := \{C \in \mathrm{GL}_r(\mathbf{R}) \mid C^t A C = A,\ B(Cc, c) < 0\ \forall c \in C_Q\}.$$

This is a subgroup of $O_A(\mathbf{R})$ of index 2.

Definition 2.10 Let

$$O_A^+(\mathbf{Z}) := O_A^+(\mathbf{R}) \cap \mathrm{GL}_r(\mathbf{Z}).$$

Remark 2.11 From $C^t A C = A$, we find $\det(C) = \pm 1$, so $O_A^+(\mathbf{Z})$ is the group of elements of $O_A^+(\mathbf{R})$ that have integer coefficients.

Remark 2.12 In some cases $O_A^+(\mathbf{Z})$ is very small. For example if $A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ then $O_A^+(\mathbf{Z})$ has only two elements: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. However, in general $O_A^+(\mathbf{Z})$ is an infinite group.

If we consider the theta functions in Definition 2.1 not only as a function of z and τ, but also as a function of c_1 and c_2, we get transformation properties with respect to $O_A^+(\mathbf{Z})$:

Proposition 2.13 *Let $C \in O_A^+(\mathbf{Z})$, $c_1, c_2 \in \overline{C}_Q$ and let $(z, \tau) \in D(c_1) \cap D(c_2)$. Let $\vartheta_A^{c_1,c_2}(z;\tau)$ be as in Definition 2.1. Then we have $C \cdot C_Q = C_Q$, $C \cdot S_Q = S_Q$, $(Cz, \tau) \in D(Cc_1) \cap D(Cc_2)$, and*

$$\vartheta^{Cc_1,Cc_2}(Cz;\tau) = \vartheta^{c_1,c_2}(z;\tau).$$

Proof: $C \cdot C_Q = C_Q$ holds by definition. If $c \in \mathbf{Z}^r$ is primitive, then Cc is also primitive. Hence we find $C \cdot S_Q = S_Q$. We have $Q(Cx) = Q(x)$ and $B(Cx, Cy) = B(x, y)$, for all $x, y \in \mathbf{R}^r$ and $C\mathbf{Z}^r = \mathbf{Z}^r$. We see

$$B\Big(c, \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\Big) = B\Big(Cc, C\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\Big) = B\Big(Cc, \frac{\mathrm{Im}(Cz)}{\mathrm{Im}(\tau)}\Big).$$

Hence if $(z, \tau) \in D(c)$ then $(Cz, \tau) \in D(Cc)$.

If we replace (c_1, c_2, z, n) by (Cc_1, Cc_2, Cz, Cn) in the definition of ϑ we get the desired transformation property. □

Remark 2.14 The C acts on both c_1 and c_2 at the same time.

Corollary 2.15 *Let $C \in O_A^+(\mathbf{Z})$, $c_1, c_2 \in \overline{C}_Q$ and let $a \in R(c_1) \cap R(c_2)$. Let $\vartheta_{a,b}^{c_1,c_2}(\tau)$ be as in Definition 2.1. Then*

$$\vartheta_{Ca,Cb}^{Cc_1,Cc_2}(\tau) = \vartheta_{a,b}^{c_1,c_2}(\tau).$$

2.5 Some examples

Example 2.16 Let $A = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, $c_1 = \left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right)$, $c_2 = \left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right)$, $e := \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$, and $a = b = \frac{1}{6}e$. Then $B(c_1, c_2) = -6$ and $Q(c_1) = Q(c_2) = -\frac{3}{2}$. If we choose C_Q such that $c_1 \in C_Q$ then also $c_2 \in C_Q$. Using (4) and (2) of Corollary 2.9, we see

$$\vartheta_{\frac{1}{6}e,\frac{1}{6}e}(\tau+1) = e^{-\frac{\pi i}{2}}\vartheta_{\frac{1}{6}e,\frac{1}{2}e}(\tau) = e^{\frac{\pi i}{6}}\vartheta_{\frac{1}{6}e,\frac{1}{6}e}(\tau). \tag{2.25}$$

Using (5) of Corollary 2.9, we see

$$\vartheta_{\frac{1}{6}e,\frac{1}{6}e}\Big(-\frac{1}{\tau}\Big) = \frac{\tau}{\sqrt{3}}\, e^{\frac{\pi i}{3}} \left(\vartheta_{-\frac{1}{6}e,-\frac{1}{6}e}(\tau) + \vartheta_{\frac{1}{6}e,-\frac{1}{6}e}(\tau) + \vartheta_{\frac{1}{2}e,-\frac{1}{6}e}(\tau)\right).$$

Using (3), (2), (3), (1) and (2) of Corollary 2.9, we see

$$\begin{aligned} \vartheta_{-\frac{1}{6}e,-\frac{1}{6}e}(\tau) &= -\vartheta_{\frac{1}{6}e,\frac{1}{6}e}(\tau) \\ \vartheta_{\frac{1}{6}e,-\frac{1}{6}e}(\tau) &= e^{-\frac{2\pi i}{3}}\vartheta_{\frac{1}{6}e,\frac{1}{6}e}(\tau) \\ \vartheta_{\frac{1}{2}e,-\frac{1}{6}e}(\tau) &= -\vartheta_{-\frac{1}{2}e,\frac{1}{6}e}(\tau) = -\vartheta_{\frac{1}{2}e,\frac{1}{6}e}(\tau) = -\vartheta_{\frac{1}{2}e,-\frac{1}{6}e}(\tau). \end{aligned}$$

Hence

$$\vartheta_{\frac{1}{6}e,\frac{1}{6}e}\Big(-\frac{1}{\tau}\Big) = -i\tau\ \vartheta_{\frac{1}{6}e,\frac{1}{6}e}(\tau). \tag{2.26}$$

We write $\rho(\nu;\tau)$ as the sum of the three expressions (2.7), (2.8) and (2.9). We will see that

$$\sum_{\nu\in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} = 0 \tag{2.27}$$

and

$$\sum_{\nu\in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_2,\nu)\Big)\beta\left(-\frac{B(c_2,\nu)^2}{Q(c_2)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} = 0. \tag{2.28}$$

To show that (2.27) holds, consider $C = \left(\begin{smallmatrix} 1 & 0 \\ -4 & -1 \end{smallmatrix}\right) \in O_A^+(\mathbf{Z})$. If we replace ν by $C\nu$ in the left hand side of (2.27) and use $Cc_1 = c_1$, we see

$$\begin{aligned} &\sum_{\nu\in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} \\ &\quad = \sum_{\nu\in C^{-1}a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,C^{-1}b)}. \end{aligned}$$

Using $C^{-1}a = a - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$, $C^{-1}b = b - \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ and $B\left(\nu, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right) = 2\nu_1 + \nu_2 \equiv \frac{1}{2} \bmod 1$ for $\nu \in a + \mathbf{Z}^2$, we see

$$\begin{aligned}
&\sum_{\nu \in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} \\
&\qquad = -\sum_{\nu \in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}.
\end{aligned}$$

Hence we get (2.27). The proof of (2.28) is similar. Here we have to use $C = \left(\begin{smallmatrix}-1 & -4\\0 & 1\end{smallmatrix}\right) \in O_A^+(\mathbf{Z})$.

Using (2.27) and (2.28) we see

$$\begin{aligned}
\vartheta_{a,b}(\tau) &= \sum_{\nu \in a+\mathbf{Z}^2} \Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big) - \operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\, e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)} \\
&= \sum_{\nu \in \left(\frac{1}{6}+\mathbf{Z}\right)^2} \Big\{\operatorname{sgn}(\nu_1) + \operatorname{sgn}(\nu_2)\Big\}\, e^{2\pi i(\frac{1}{2}\nu_1^2 + 2\nu_1\nu_2 + \frac{1}{2}\nu_2^2)\tau + \pi i(\nu_1+\nu_2)} \\
&= 2e^{\frac{\pi i}{3}} q^{\frac{1}{12}} \Big(\sum_{n,m\geq 0} - \sum_{n,m<0}\Big)(-1)^{n+m} q^{\frac{1}{2}n^2 + 2nm + \frac{1}{2}m^2 + \frac{1}{2}n + \frac{1}{2}m},
\end{aligned} \tag{2.29}$$

where we have substituted $\nu_1 = \frac{1}{6} + n$ and $\nu_2 = \frac{1}{6} + m$ in the last step.

From (2.29) together with (2.25) and (2.26) we see that $\vartheta_{a,b}$ is a holomorphic modular form of weight 1, with the same transformation properties as η^2. Hence $\vartheta_{a,b}$ is a multiple of η^2 (their quotient is a holomorphic function on the compact Riemann surface $\mathcal{H}/\operatorname{SL}_2(\mathbf{Z}) \cup \{\infty\}$). By comparing the first Fourier coefficients we find

$$\vartheta_{\frac{1}{6}e,\frac{1}{6}e} = 2e^{\frac{\pi i}{3}}\eta^2,$$

or equivalently

$$\Big(\sum_{n,m\geq 0} - \sum_{n,m<0}\Big)(-1)^{n+m} q^{\frac{1}{2}n^2 + 2nm + \frac{1}{2}m^2 + \frac{1}{2}n + \frac{1}{2}m} = (q)_\infty^2.$$

Example 2.17 This example is similar to the previous one, so some of the details are omitted.

Let $A = \left(\begin{smallmatrix}1 & 0\\0 & -3\end{smallmatrix}\right)$, $c_1 = \left(\begin{smallmatrix}-3\\2\end{smallmatrix}\right)$, $c_2 = \left(\begin{smallmatrix}3\\2\end{smallmatrix}\right)$ and $a = b = \frac{1}{6}\left(\begin{smallmatrix}3\\-1\end{smallmatrix}\right)$. Then $B(c_1,c_2) = -21$ and $Q(c_1) = Q(c_2) = -\frac{3}{2}$. Using Corollary 2.9, we see

$$\vartheta_{a,b}(\tau+1) = e^{\frac{\pi i}{6}}\vartheta_{a,b}(\tau)$$

and

$$\vartheta_{a,b}\Big(-\frac{1}{\tau}\Big) = -i\tau\ \vartheta_{a,b}(\tau).$$

We write $\rho(\nu;\tau)$ as the sum of the three expressions (2.7), (2.8) and (2.9). We have

$$\sum_{\nu\in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_1,\nu)\Big)\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)}=0$$

and

$$\sum_{\nu\in a+\mathbf{Z}^2} \operatorname{sgn}\Big(B(c_2,\nu)\Big)\beta\left(-\frac{B(c_2,\nu)^2}{Q(c_2)}y\right) e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)}=0.$$

To get the first equation we use $C=\left(\begin{smallmatrix}-7 & -12\\ 4 & 7\end{smallmatrix}\right)\in O_A^+(\mathbf{Z})$ ($Cc_1=c_1$, $C^{-1}a=a+\left(\begin{smallmatrix}-2\\ 1\end{smallmatrix}\right)$ and $C^{-1}b=b+\left(\begin{smallmatrix}-2\\ 1\end{smallmatrix}\right)$). To get the second equation we use $C=\left(\begin{smallmatrix}-7 & 12\\ -4 & 7\end{smallmatrix}\right)\in O_A^+(\mathbf{Z})$ ($Cc_2=c_2$, $C^{-1}a=a-\left(\begin{smallmatrix}4\\ 3\end{smallmatrix}\right)$ and $C^{-1}b=b-\left(\begin{smallmatrix}4\\ 3\end{smallmatrix}\right)$). Hence we see

$$\begin{aligned}\vartheta_{a,b}(\tau) &= \sum_{\nu\in a+\mathbf{Z}^2}\Big\{\operatorname{sgn}\Big(B(c_1,\nu)\Big)-\operatorname{sgn}\Big(B(c_2,\nu)\Big)\Big\}\, e^{2\pi i Q(\nu)\tau+2\pi i B(\nu,b)}\\ &= -\sum_{\nu\in\frac{1}{6}\left(\begin{smallmatrix}3\\ -1\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(\nu_1+2\nu_2)+\operatorname{sgn}(\nu_1-2\nu_2)\Big\}\, e^{2\pi i(\frac{1}{2}\nu_1^2-\frac{3}{2}\nu_2^2)\tau+\pi i(\nu_1+\nu_2)}\\ &= -2e^{\frac{\pi i}{3}}q^{\frac{1}{12}}\Bigg(\sum_{n+2m,n-2m\geq 0}-\sum_{n+2m,n-2m<0}\Bigg)(-1)^{n+m}q^{\frac{1}{2}n^2-\frac{3}{2}m^2+\frac{1}{2}n+\frac{1}{2}m},\end{aligned}$$

where we have substituted $\nu_1=\frac{1}{2}+n$ and $\nu_2=-\frac{1}{6}+m$ in the last step. Replacing n by $-n-1$, we see

$$\begin{aligned}&\sum_{n+2m,n-2m<0}(-1)^{n+m}q^{\frac{1}{2}n^2-\frac{3}{2}m^2+\frac{1}{2}n+\frac{1}{2}m}\\ &\qquad=\sum_{n+2m,n-2m\geq 0}(-1)^{n+m}q^{\frac{1}{2}n^2-\frac{3}{2}m^2+\frac{1}{2}n+\frac{1}{2}m},\end{aligned}$$

so

$$\vartheta_{a,b}(\tau)=-4e^{\frac{\pi i}{3}}q^{\frac{1}{12}}\sum_{n\geq 2|m|}(-1)^{n+m}q^{\frac{1}{2}n^2-\frac{3}{2}m^2+\frac{1}{2}n+\frac{1}{2}m}.$$

We see that $\vartheta_{a,b}$ is a holomorphic modular form of weight 1, with the same transformation properties as η^2. Hence $\vartheta_{a,b}$ is a multiple of η^2. By comparing the first Fourier coefficients we find

$$\vartheta_{a,b}=-4e^{\frac{\pi i}{3}}\eta^2,$$

or equivalently

$$\sum_{n\geq 2|m|}(-1)^{n+m}q^{\frac{1}{2}n^2-\frac{3}{2}m^2+\frac{1}{2}n+\frac{1}{2}m}=(q)_\infty^2.$$

This last equation is proven in [1, pp. 451], using different techniques. In that article several similar results are proven. The modular transformation properties of the functions involved can be found using the same method as in the examples presented here.

These examples are very special: in general, $\vartheta_{a,b}$ is not a holomorphic function. However, for the special values of c_1, c_2, a and b given here, $\vartheta_{a,b}$ is holomorphic.

In [20] a theorem about the modularity of a certain family of q-series associated with indefinite binary quadratic forms is given. This result may also be found using the same method as in the examples presented here.

In the next two chapters, we will see some other examples. In these examples, the ϑ-functions are not holomorphic.