

Chapter 3

Fourier Coefficients of Meromorphic Jacobi Forms

3.1 Introduction

In this chapter we consider functions $\varphi : \mathbf{C} \times \mathcal{H} \rightarrow \mathbf{C}$ that satisfy

$$\varphi(z + \lambda\tau + \mu; \tau) = e^{-2\pi im(\lambda^2\tau + 2\lambda z)}\varphi(z; \tau) \quad \forall \lambda, \mu \in \mathbf{Z} \quad (\text{E})$$

and

$$\varphi\left(\frac{z}{c\tau + d}; \frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k e^{2\pi imcz^2/(c\tau + d)}\varphi(z; \tau) \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbf{Z}), \quad (\text{M})$$

with $k \in \mathbf{Z}$ and $m \in \mathbf{Z}_{>0}$. The first equation gives the transformation law with respect to $z \mapsto z + \lambda\tau + \mu$ and will be denoted by (E), for elliptic. The second equation gives the transformation law with respect to $\text{SL}_2(\mathbf{Z})$ and will be denoted by (M), for modular. Jacobi forms of weight k and index m satisfy both (E) and (M).

It is a classical result, see [9, pp. 57–59], that the space of Jacobi forms of weight k and index m is isomorphic to a certain space of (vector-valued) modular forms of weight $k - \frac{1}{2}$ in one variable:

Theorem 3.1 *If φ is holomorphic as a function of z and satisfies (E), we have*

$$\varphi(z; \tau) = \sum_{l \bmod 2m} h_l(\tau)\vartheta_{m,l}(z; \tau), \quad (3.1)$$

with Fourier coefficients

$$h_l(\tau) = e^{-\pi il^2\tau/2m} \int_p^{p+1} \varphi(z; \tau) e^{-2\pi ilz} dz \quad p \in \mathbf{C}$$

and

$$\vartheta_{m,l}(z; \tau) = \sum_{\substack{\lambda \in \mathbf{Z} \\ \lambda \equiv l \pmod{2m}}} e^{\pi i \lambda^2 \tau / 2m + 2\pi i \lambda z}. \quad (3.2)$$

If φ also satisfies the transformation (M), then we have for each l :

$$h_l(\tau + 1) = e^{-\pi i l^2 / 2m} h_l(\tau) \quad (3.3)$$

and

$$h_l\left(-\frac{1}{\tau}\right) = \frac{\tau^k}{\sqrt{-i\tau}} \frac{1}{\sqrt{2m}} \sum_{\nu \pmod{2m}} e^{\pi i \nu / m} h_\nu(\tau). \quad (3.4)$$

Proof: If we take $\lambda = 0$ and $\mu = 1$ in (E) we see that φ is 1-periodic. Hence we can write

$$\varphi(z; \tau) = \sum_{r \in \mathbf{Z}} h_r(\tau) e^{\pi i r^2 \tau / 2m + 2\pi i r z}, \quad (3.5)$$

with

$$h_r(\tau) = e^{-\pi i r^2 \tau / 2m} \int_p^{p+1} \varphi(z; \tau) e^{-2\pi i r z} dz \quad p \in \mathbf{C}.$$

(The extra factor $e^{\pi i r^2 \tau / 2m}$ in the Fourier coefficients is for convenience).

If we use (E) with $\lambda = 1$ and $\mu = 0$ we see that $h_{r+2m} = h_r$. Hence h_r depends only on $r \pmod{2m}$. Putting this into (3.5) gives

$$\begin{aligned} \varphi(z; \tau) &= \sum_{r \in \mathbf{Z}} h_r(\tau) e^{\pi i r^2 \tau / 2m + 2\pi i r z} = \sum_{l \pmod{2m}} \sum_{\substack{\lambda \in \mathbf{Z} \\ \lambda \equiv l \pmod{2m}}} h_\lambda(\tau) e^{\pi i \lambda^2 \tau / 2m + 2\pi i \lambda z} \\ &= \sum_{l \pmod{2m}} h_l(\tau) \vartheta_{m,l}(z; \tau), \end{aligned}$$

with $\vartheta_{m,l}$ as in (3.2).

If φ also satisfies the transformation (M), then we get the transformation properties (3.3) and (3.4) of h_l from the transformation properties of $\vartheta_{m,l}$ and the decomposition given in (3.1); see [9, pp. 58–59] for details. \square

The h_l are more or less the Fourier coefficients of φ , if we consider φ as a function of z . These Fourier coefficients form a vector-valued modular form.

In [3] Andrews gives most of the fifth order mock theta functions as Fourier coefficients of meromorphic Jacobi forms (i.e. meromorphic as a function of z), namely certain quotients of ordinary Jacobi theta-series. In this chapter (see Theorem 3.9), we generalize Theorem 3.1 to include meromorphic Jacobi forms. We give the result only for Jacobi forms on the full Jacobi group (i.e. satisfying (E) and (M) without any congruence restrictions on (λ, μ) or $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$), but it could certainly be generalized to congruence subgroups (and vector-valued Jacobi forms) and could then be combined with Andrews's identities to obtain information about the modular properties of the fifth order mock theta functions. We will not carry this out, since the same results will be obtained in Chapter 4 using instead the results on indefinite ϑ -functions from Chapter 2.

3.2 A building block

In this section we define functions $f_u : \mathbf{C} \times \mathcal{H} \rightarrow \mathbf{C}$ and $\tilde{f}_u : \mathbf{C} \times \mathcal{H} \rightarrow \mathbf{C}$, which will be used in the next section as building blocks for meromorphic Jacobi forms.

Definition 3.2 Let $u \in \mathbf{C}$ and $m \in \mathbf{Z}_{>0}$. Define $f_u : \mathbf{C} \times \mathcal{H} \rightarrow \mathbf{C}$ by

$$f_u(z; \tau) = f_u^{(m)}(z; \tau) := \sum_{\lambda \in \mathbf{Z}} \frac{e^{2\pi i m \lambda^2 \tau + 4\pi i m \lambda z}}{1 - e^{2\pi i \lambda \tau + 2\pi i (z-u)}}.$$

Note the similarity of this sum with the Lerch sums studied in Chapter 1. The function $f^{(1/2)}$ is the sum studied in Section 1.2. The following result is the analogue of Proposition 1.4 and Proposition 1.5.

Proposition 3.3 *We have*

- (1) f_u satisfies (E),
- (2) $z \mapsto f_u(z; \tau)$ is a meromorphic function, with simple poles in $u + \mathbf{Z}\tau + \mathbf{Z}$, and residue $-\frac{1}{2\pi i}$ in $z = u$,
- (3) $f_{u+1}(z; \tau) = f_u(z; \tau)$
- (4) $f_u(z; \tau) - e^{-2\pi i m \tau - 4\pi i m u} f_{u+\tau}(z; \tau) = \sum_{l=0}^{2m-1} e^{-\pi i l^2 \tau / 2m - 2\pi i l u} \vartheta_{m,l}(z; \tau)$,
- (5) $f_u(z; \tau + 1) = f_u(z; \tau)$,
- (6) $f_u(z; \tau) - \frac{1}{\tau} e^{2\pi i m (u^2 - z^2) / \tau} f_{\frac{u}{\tau}}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \sum_{l=0}^{2m-1} h_l(u; \tau) \vartheta_{m,l}(z; \tau)$, with

$$h_l(u; \tau) = i e^{-\pi i l^2 \tau / 2m - 2\pi i l u} \int_L \frac{e^{2\pi i m \tau x^2 - 2\pi (2mu + l\tau)x}}{1 - e^{2\pi x}} dx,$$

where $L = \mathbf{R} - it$ with $0 < t < 1$. This path can be deformed into the real axis indented by the lower half of a small circle with the origin as its centre.

Proof: We see immediately that the series converges absolutely, unless $z = u - \lambda\tau + \mu$ for some $\lambda, \mu \in \mathbf{Z}$, in which case one term in the sum becomes infinite. Hence $z \mapsto f_u(z; \tau)$ is meromorphic, with simple poles only in the points $z = u - \lambda\tau + \mu$ ($\lambda, \mu \in \mathbf{Z}$).

(1) It is easy to see that $f_u(z + \mu) = f_u(z)$ for all $\mu \in \mathbf{Z}$. Also

$$\begin{aligned} f_u(z + \mu\tau; \tau) &= \sum_{\lambda \in \mathbf{Z}} \frac{e^{2\pi i m \lambda^2 \tau + 4\pi i m \lambda z + 4\pi i m \mu \lambda \tau}}{1 - e^{2\pi i \lambda \tau + 2\pi i (z-u) + 2\pi i \mu \tau}} \\ &= e^{-2\pi i m \mu^2 \tau - 4\pi i m \mu z} \sum_{\lambda \in \mathbf{Z}} \frac{e^{2\pi i m (\lambda + \mu)^2 \tau + 4\pi i m (\lambda + \mu) z}}{1 - e^{2\pi i (\lambda + \mu) \tau + 2\pi i (z-u)}} \\ &= e^{-2\pi i m \mu^2 \tau - 4\pi i m \mu z} f_u(z; \tau). \end{aligned}$$

(2) We have already seen that $z \mapsto f_u(z; \tau)$ is a meromorphic function, with simple poles in $u + \mathbf{Z}\tau + \mathbf{Z}$. The pole in $z = u$ comes from the term $\lambda = 0$. We see

$$\lim_{z \rightarrow u} (z - u)f_u(z; \tau) = \lim_{z \rightarrow u} \frac{(z - u)}{1 - e^{2\pi i(z-u)}} = -\frac{1}{2\pi i}.$$

(3) Trivial.

(4) If we replace λ by $\lambda + 1$ in the definition we find

$$e^{-2\pi i m \tau - 4\pi i m u} f_{u+\tau}(z; \tau) = \sum_{\lambda \in \mathbf{Z}} \frac{e^{2\pi i m \lambda^2 \tau + 4\pi i m \lambda z + 4\pi i m \lambda \tau + 4\pi i m (z-u)}}{1 - e^{2\pi i \lambda \tau + 2\pi i (z-u)}}.$$

Hence

$$\begin{aligned} f_u(z; \tau) - e^{-2\pi i m \tau - 4\pi i m u} f_{u+\tau}(z; \tau) &= \sum_{\lambda \in \mathbf{Z}} e^{2\pi i m \lambda^2 \tau + 4\pi i m \lambda z} \frac{1 - e^{4\pi i m \lambda \tau + 4\pi i m (z-u)}}{1 - e^{2\pi i \lambda \tau + 2\pi i (z-u)}} \\ &= \sum_{\lambda \in \mathbf{Z}} e^{2\pi i m \lambda^2 \tau + 4\pi i m \lambda z} \sum_{l=0}^{2m-1} e^{2\pi i l \lambda \tau + 2\pi i l (z-u)} \\ &= \sum_{l=0}^{2m-1} e^{-\pi i l^2 \tau / 2m - 2\pi i l u} \vartheta_{m,l}(z; \tau). \end{aligned}$$

In the last step we have changed the order of summation and substituted $\mu = 2m\lambda + l$.

(5) Trivial.

(6) If $z \mapsto f(z; \tau)$ satisfies (E), then so does $z \mapsto e^{-2\pi i m z^2 / \tau} f\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)$. The function $z \mapsto e^{-2\pi i m z^2 / \tau} f_{\frac{u}{\tau}}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)$ is meromorphic, with simple poles in $u + \mathbf{Z}\tau + \mathbf{Z}$, and residue $-\frac{\tau}{2\pi i} e^{-2\pi i m u^2 / \tau}$ in $z = u$. So

$$z \mapsto f_u(z; \tau) - \frac{1}{\tau} e^{2\pi i m (u^2 - z^2) / \tau} f_{\frac{u}{\tau}}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right)$$

is a holomorphic function, which satisfies (E). Theorem 3.1 shows that there are h_l such that

$$f_u(z; \tau) - \frac{1}{\tau} e^{2\pi i m (u^2 - z^2) / \tau} f_{\frac{u}{\tau}}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) = \sum_{l \bmod 2m} h_l(\tau) \vartheta_{m,l}(z; \tau).$$

If we restrict z by $0 < \text{Im}(u - z) < \text{Im}(\tau)$ and expand $\left(1 - e^{2\pi i \lambda \tau + 2\pi i (z-u)}\right)^{-1}$ into a geometric series, we see

$$\begin{aligned} f_u(z; \tau) &= \sum_{\lambda \in \mathbf{Z}} e^{2\pi i m \lambda^2 \tau + 4\pi i m \lambda z} \text{sgn}\left(\lambda - \frac{1}{2}\right) \sum_{\substack{\mu \in \mathbf{Z} \\ \text{sgn}(\lambda - \frac{1}{2}) = \text{sgn}(\mu + \frac{1}{2})}} e^{2\pi i \lambda \mu \tau + 2\pi i (z-u)\mu} \\ &= \sum_{\substack{\lambda, \mu \in \mathbf{Z} \\ \text{sgn}(\lambda - \frac{1}{2}) = \text{sgn}(\mu + \frac{1}{2})}} \text{sgn}\left(\lambda - \frac{1}{2}\right) e^{2\pi i m \lambda^2 \tau + 2\pi i \lambda \mu \tau - 2\pi i \mu u + 2\pi i (2m\lambda + \mu)z}. \end{aligned}$$

But if $\operatorname{sgn}(\lambda - \frac{1}{2}) = \operatorname{sgn}(\mu + \frac{1}{2})$ then $2m\lambda + \mu$ is either $\geq 2m$ or < 0 , so

$$\int_p^{p+1} f_u(z; \tau) e^{-2\pi i l z} dz = 0,$$

for $0 \leq l \leq 2m-1$, for any p satisfying $0 < \operatorname{Im}(u-p) < \operatorname{Im}(\tau)$. Hence for $0 \leq l \leq 2m-1$ we find

$$\begin{aligned} h_l(\tau) &= -\frac{1}{\tau} e^{2\pi i m u^2 / \tau - \pi i l^2 \tau / 2m} \int_p^{p+1} e^{-2\pi i m z^2 / \tau - 2\pi i l z} f_{\frac{u}{\tau}}\left(\frac{z}{\tau}; -\frac{1}{\tau}\right) dz \\ &= -\frac{1}{\tau} e^{2\pi i m u^2 / \tau - \pi i l^2 \tau / 2m} \int_p^{p+1} \sum_{\lambda \in \mathbf{Z}} \frac{e^{-2\pi i m (z-\lambda)^2 / \tau - 2\pi i l (z-\lambda)}}{1 - e^{2\pi i (z-\lambda-u) / \tau}} dz \\ &= -\frac{1}{\tau} e^{2\pi i m u^2 / \tau - \pi i l^2 \tau / 2m} \int_{p+\mathbf{R}} \frac{e^{-2\pi i m z^2 / \tau - 2\pi i l z}}{1 - e^{2\pi i (z-u) / \tau}} dz. \end{aligned}$$

If we substitute $x = i(z-u)/\tau$ in this last integral and use Cauchy's theorem, we get the desired result. \square

We see that the transformation law with respect to $\operatorname{SL}_2(\mathbf{Z})$ for f_u is rather complicated. However, if we modify the definition a little (in Theorem 1.11 we did something similar), we get a function which is no longer holomorphic as a function of τ , but has simpler transformation properties with respect to $\operatorname{SL}_2(\mathbf{Z})$.

Definition 3.4 Let $u \in \mathbf{C}$. Define $\tilde{f}_u : \mathbf{C} \times \mathcal{H} \rightarrow \mathbf{C}$ by

$$\tilde{f}_u(z; \tau) = f_u(z; \tau) - \frac{1}{2} \sum_{l \bmod 2m} R_{m,l}(u; \tau) \vartheta_{m,l}(z; \tau),$$

with

$$R_{m,l}(u; \tau) = \sum_{\substack{\lambda \in \mathbf{Z} \\ \lambda \equiv l \pmod{2m}}} \left\{ \operatorname{sgn}\left(\lambda + \frac{1}{2}\right) - E\left((\lambda + 2m \operatorname{Im}(u)/y) \sqrt{y/m}\right) \right\} e^{-\pi i \lambda^2 \tau / 2m - 2\pi i \lambda u},$$

$y = \operatorname{Im}(\tau)$ and E as in Definition 1.6.

The function $R_{m,l}$ is the analogue of the function R defined in Lemma 1.8. We will not show that the series defining $R_{m,l}$ converges, since this is similar to the convergence of R , proven in Lemma 1.8.

Proposition 3.5 *We have*

(1) $\tilde{f}_u(z; \tau)$ transforms like a 2-variable Jacobi form of weight 1 and index $\begin{pmatrix} 2m & 0 \\ 0 & -2m \end{pmatrix}$ with respect to $(z, u, \tau) \in \mathbf{C}^2 \times \mathcal{H}$, i.e.

(a) \tilde{f}_u satisfies (E),

(b) $\tilde{f}_{u+\lambda\tau+\mu}(z; \tau) = e^{2\pi im(\lambda^2\tau+2\lambda u)} \tilde{f}_u(z; \tau)$ for all $\lambda, \mu \in \mathbf{Z}$,

(c)

$$\tilde{f}_{\frac{u}{c\tau+d}} \left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d} \right) = (c\tau+d) e^{2\pi imc(z^2-u^2)/(c\tau+d)} \tilde{f}_u(z; \tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$.

(2) for fixed u , $z \mapsto \tilde{f}_u(z; \tau)$ is a meromorphic function, with simple poles in $u + \mathbf{Z}\tau + \mathbf{Z}$, and no other poles, and residue $-\frac{1}{2\pi i}$ in $z = u$,

(3) \tilde{f} can be seen as a indefinite ϑ -series (see Definition 2.1), namely

$$\tilde{f}_u(z; \tau) = \frac{1}{2} \vartheta_A^{c_1, c_2} \left(\begin{pmatrix} z-u \\ 2mu \end{pmatrix}; \tau \right),$$

with $A = \begin{pmatrix} 2m & 1 \\ 1 & 0 \end{pmatrix}$, $c_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in S_Q$ and $c_2 = \begin{pmatrix} -1 \\ 2m \end{pmatrix} \in C_Q$.

Proof: (2) This follows directly from (2) of Proposition 3.3 and the fact that $\vartheta_{m,l}$ is holomorphic.

(3) Using the geometric series expansion we see

$$\frac{1}{1 - e^{2\pi i\lambda\tau+2\pi i(z-u)}} = \frac{1}{2} \sum_{\mu \in \mathbf{Z}} \left\{ \mathrm{sgn} \left(\lambda + \mathrm{Im}(z-u)/y \right) + \mathrm{sgn} \left(\mu + \frac{1}{2} \right) \right\} e^{2\pi i\lambda\mu\tau+2\pi i(z-u)\mu}.$$

Hence

$$\begin{aligned} f_u(z; \tau) &= \sum_{\lambda, \mu \in \mathbf{Z}} \frac{1}{2} \left\{ \mathrm{sgn} \left(\lambda + \mathrm{Im}(z-u)/y \right) + \mathrm{sgn} \left(\mu + \frac{1}{2} \right) \right\} e^{2\pi i(m\lambda^2+\lambda\mu)\tau+2\pi i(2mz\lambda+(z-u)\mu)} \\ &= \sum_{n \in \mathbf{Z}^2} \frac{1}{2} \left\{ \mathrm{sgn} \left(n_1 + \mathrm{Im}(z-u)/y \right) + \mathrm{sgn} \left(n_2 + \frac{1}{2} \right) \right\} e^{2\pi iQ(n)\tau+2\pi iB \left(n, \begin{pmatrix} z-u \\ 2mu \end{pmatrix} \right)}. \end{aligned}$$

We also have

$$\begin{aligned} &\sum_{l \bmod 2m} R_{m,l}(u; \tau) \vartheta_{m,l}(z; \tau) \\ &= \sum_{\substack{\lambda, \mu \in \mathbf{Z} \\ \lambda \equiv \mu \bmod 2m}} \left\{ \mathrm{sgn} \left(\lambda + \frac{1}{2} \right) - E \left((\lambda + 2m \mathrm{Im} u/y) \sqrt{y/m} \right) \right\} e^{\pi i(\mu^2-\lambda^2)\tau/2m+2\pi i(\mu z-\lambda u)}. \end{aligned}$$

If we substitute $\lambda = n_2$ and $\mu = 2mn_1 + n_2$, we find

$$\begin{aligned} &\sum_{l \bmod 2m} R_{m,l}(u; \tau) \vartheta_{m,l}(z; \tau) \\ &= \sum_{n \in \mathbf{Z}^2} \left\{ \mathrm{sgn} \left(n_2 + \frac{1}{2} \right) - E \left((n_2 + 2m \mathrm{Im} u/y) \sqrt{y/m} \right) \right\} e^{2\pi iQ(n)\tau+2\pi iB \left(n, \begin{pmatrix} z-u \\ 2mu \end{pmatrix} \right)}. \end{aligned}$$

Hence

$$\begin{aligned}
\tilde{f}_u(z; \tau) &= f_u(z; \tau) - \frac{1}{2} \sum_{l \bmod 2m} \overline{R_{m,l}(u; \tau)} \vartheta_{m,l}(z; \tau) \\
&= \sum_{n \in \mathbf{Z}^2} \frac{1}{2} \left\{ \operatorname{sgn}(n_1 + \operatorname{Im}(z-u)/y) + E\left((n_2 + 2m \operatorname{Im} u/y) \sqrt{y/m}\right) \right\} \\
&\quad \cdot e^{2\pi i Q(n)\tau + 2\pi i B\left(n, \begin{pmatrix} z-u \\ 2mu \end{pmatrix}\right)} \\
&= \frac{1}{2} \vartheta_A^{c_1, c_2} \left(\begin{pmatrix} z-u \\ 2mu \end{pmatrix}; \tau \right),
\end{aligned}$$

which proves (3).

(1a) and (1b) follow directly from (3) and the transformation properties of $\vartheta_A^{c_1, c_2}$ given in (2) of Proposition 2.7.

(1c) Using (3) and the transformation properties of $\vartheta_A^{c_1, c_2}$ given in (6) and (7) of Proposition 2.7 we see

$$\tilde{f}_u(z; \tau + 1) = \tilde{f}_u(z; \tau) \quad (3.6)$$

and

$$\tilde{f}_{\frac{z}{\tau}} \left(\frac{z}{\tau}; -\frac{1}{\tau} \right) = \tau e^{2\pi i m(z^2 - u^2)/\tau} \tilde{f}_u(z; \tau). \quad (3.7)$$

Combining these results we get (1c). \square

Remark 3.6 We do not need (3) to prove (1): We could also prove (3.7) using (6) of Proposition 3.3 and an analogue of (2) of Proposition 1.10 for $R_{m,l}$:

$$R_{m,l}(u; \tau) + \frac{i}{\sqrt{-i\tau}} e^{2\pi i m u^2/\tau} \frac{1}{\sqrt{2m}} \sum_{\nu \bmod 2m} e^{-\pi i l \nu/m} R_{m,\nu} \left(\frac{u}{\tau}; -\frac{1}{\tau} \right) = 2h_l(u; \tau).$$

We will not prove this equation. Part (1b) and equation (3.6) may also be proved by using properties of f_u given in Proposition 3.3 and properties of $R_{m,l}$, which we will not give here. Part (1a) follows directly from the fact that both f_u and $\vartheta_{m,l}$ satisfy (E).

Proposition 3.7 *Let $R_{m,l}$ be as in Definition 3.4. Then*

(1) *if $\alpha, \beta \in \mathbf{R}$ then*

$$\frac{\partial}{\partial \tau} e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha\tau + \beta; \tau) = -i \sqrt{\frac{m}{y}} e^{4\pi i m \alpha \beta} \sum_{\lambda \in \alpha + \frac{1}{2m} + \mathbf{Z}} \lambda e^{-2\pi i m \lambda^2 \tau - 4\pi i m \lambda \beta},$$

(2) $\tau \mapsto e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha\tau + \beta; \tau)$ *is an eigenfunction of the weight 1/2 Casimir operator $\Omega_{\frac{1}{2}} = -4y^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} + iy \frac{\partial}{\partial \bar{\tau}} + \frac{3}{16}$ with eigenvalue $\frac{3}{16}$.*

Proof: (1) We have

$$\begin{aligned}
& \frac{\partial}{\partial \bar{\tau}} e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha \tau + \beta; \tau) \\
&= \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \cdot \\
& \quad \sum_{\substack{\lambda \in \mathbf{Z} \\ \lambda \equiv l \pmod{2m}}} \left\{ \operatorname{sgn} \left(\lambda + \frac{1}{2} \right) - E \left((\lambda + 2m\alpha) \sqrt{y/m} \right) \right\} e^{-2\pi i m (\lambda/2m + \alpha)^2 \tau - 2\pi i \lambda \beta} \\
&= -\frac{i}{2} \sqrt{\frac{m}{y}} \sum_{\substack{\lambda \in \mathbf{Z} \\ \lambda \equiv l \pmod{2m}}} \left(\frac{\lambda}{2m} + \alpha \right) E' \left((\lambda + 2m\alpha) \sqrt{y/m} \right) e^{-2\pi i m (\lambda/2m + \alpha)^2 \tau - 2\pi i \lambda \beta} \\
&= -i \sqrt{\frac{m}{y}} \sum_{\substack{\lambda \in \mathbf{Z} \\ \lambda \equiv l \pmod{2m}}} \left(\frac{\lambda}{2m} + \alpha \right) e^{-2\pi i m (\lambda/2m + \alpha)^2 \tau - 2\pi i \lambda \beta}.
\end{aligned}$$

If we now substitute $\lambda' = \frac{\lambda}{2m} + \alpha$ we get the desired result.

(2) From (1) we see that $\tau \mapsto \sqrt{y} \frac{\partial}{\partial \bar{\tau}} e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha \tau + \beta; \tau)$ is anti-holomorphic, so

$$\frac{\partial}{\partial \tau} \sqrt{y} \frac{\partial}{\partial \bar{\tau}} e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha \tau + \beta; \tau) = 0$$

We can write the operator $\Omega_{\frac{1}{2}} = -4y^2 \frac{\partial^2}{\partial \tau \partial \bar{\tau}} + iy \frac{\partial}{\partial \bar{\tau}} + \frac{3}{16}$ as

$$\Omega_{\frac{1}{2}} = \frac{3}{16} - 4y^{3/2} \frac{\partial}{\partial \tau} \sqrt{y} \frac{\partial}{\partial \bar{\tau}}.$$

Hence

$$\Omega_{\frac{1}{2}} e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha \tau + \beta; \tau) = \frac{3}{16} e^{-2\pi i m \alpha^2 \tau} R_{m,l}(\alpha \tau + \beta; \tau),$$

which proves (2). □

3.3 Transformation properties

Before we can state the main result we need the following

Definition 3.8 Let $u \in \mathbf{C}$ and let f be a real-analytic function in a neighbourhood of u . If g is a meromorphic function with a pole of order s in u , then $f \cdot g$ has, in a neighbourhood of u , an expansion

$$\sum_{n \geq -s} \sum_{m \geq 0} a_{nm} (v - u)^n (\bar{v} - \bar{u})^m.$$

We define

$$\operatorname{Res}_{v=u} [f(v)g(v)] = a_{-1,0} = \frac{1}{(s-1)!} \left. \frac{\partial^{s-1}}{\partial v^{s-1}} \right|_{v=u} (f(v) \cdot (v-u)^s g(v)).$$

If f is holomorphic the definition coincides with the usual definition of the residue.

Now the main result:

Theorem 3.9 *Let φ be such that $z \mapsto \varphi(z; \tau)$, for fixed $\tau \in \mathcal{H}$, is a meromorphic function. If φ satisfies (E), then φ has a development of the form*

$$\varphi(z; \tau) = \sum_{l \bmod 2m} h_l(\tau) \vartheta_{m,l}(z; \tau) - 2\pi i \sum_{u \in \operatorname{Sing} \varphi(\cdot; \tau) \bmod \Lambda_\tau} \operatorname{Res}_{v=u} [\tilde{f}_v(z; \tau) \varphi(v; \tau)],$$

for $\tau \in \mathcal{H}$ and $z \notin \operatorname{Sing} \varphi(\cdot; \tau)$, with

$$\operatorname{Sing} \varphi(\cdot; \tau) := \{u \in \mathbf{C} \mid z \mapsto \varphi(z; \tau) \text{ has a pole in } u\},$$

$$\Lambda_\tau = \mathbf{Z}\tau + \mathbf{Z},$$

and, for $0 \leq l \leq 2m-1$ and any $p \in \mathbf{C}$ such that there are no poles on the boundary ∂P_p of $P_p := p + (0, 1)\tau + (0, 1)$,

$$h_l(\tau) = e^{-\pi i l^2 \tau / 2m} \int_p^{p+1} \varphi(z; \tau) e^{-2\pi i l z} dz - \pi i \sum_{u \in \operatorname{sing}_p \varphi(\cdot; \tau)} \operatorname{Res}_{v=u} [R_{m,l}(v; \tau) \varphi(v; \tau)],$$

with $\operatorname{sing}_p \varphi(\cdot; \tau) = \operatorname{Sing} \varphi(\cdot; \tau) \cap P_p$.

If φ also satisfies (M), then the vector $(h_l)_{l \bmod 2m}$ transforms under the action of $\operatorname{SL}_2(\mathbf{Z})$ as in equations (3.3) and (3.4).

Proof: Let $p \in \mathbf{C}$ be such that $z \mapsto \varphi(z; \tau)$ has no poles on ∂P_p . Let $z \in P_p$, $z \notin \operatorname{Sing} \varphi(\cdot; \tau)$. Now consider

$$\int_{\partial P_p} f_v(z; \tau) \varphi(v; \tau) dv.$$

We compute this integral in two different ways. On the one hand, the function we are integrating is 1-periodic. Hence

$$\begin{aligned} & \int_{\partial P_p} f_v(z; \tau) \varphi(v; \tau) dv \\ &= \int_p^{p+1} f_v(z; \tau) \varphi(v; \tau) dv - \int_p^{p+1} f_{v+\tau}(z; \tau) \varphi(v+\tau; \tau) dv \\ &= \int_p^{p+1} (f_v(z; \tau) - e^{-2\pi i m \tau - 4\pi i m v} f_{v+\tau}(z; \tau)) \varphi(v; \tau) dv \\ &= \sum_{l=0}^{2m-1} e^{-\pi i l^2 \tau / 2m} \vartheta_{m,l}(z; \tau) \int_p^{p+1} \varphi(v; \tau) e^{-2\pi i l v} dv, \end{aligned}$$

by (4) of Proposition 3.3. On the other hand, we can compute the integral using the residue theorem. The poles of $v \mapsto f_v(z; \tau)\varphi(v; \tau)$ inside P_p are the poles of φ inside P_p , together with z , and the residue in $v = z$ is $\frac{1}{2\pi i}\varphi(z; \tau)$. Hence

$$\begin{aligned} & \int_{\partial P_p} f_v(z; \tau)\varphi(v; \tau)dv \\ &= 2\pi i \sum_{u \in \text{Sing}_p \varphi(\cdot; \tau)} \text{Res}_{v=u} \left[f_v(z; \tau)\varphi(v; \tau) \right] + \varphi(z; \tau) \\ &= 2\pi i \sum_{u \in \text{Sing}_p \varphi(\cdot; \tau)} \text{Res}_{v=u} \left[\tilde{f}_v(z; \tau)\varphi(v; \tau) \right] + \varphi(z; \tau) \\ & \quad + \pi i \sum_{l \bmod 2m} \vartheta_{m,l}(z; \tau) \sum_{u \in \text{Sing}_p \varphi(\cdot; \tau)} \text{Res}_{v=u} \left[R_{m,l}(v; \tau)\varphi(v; \tau) \right] \end{aligned}$$

If we compare the two evaluations of the integral, we obtain

$$\varphi(z; \tau) = \sum_{l \bmod 2m} h_l(\tau)\vartheta_{m,l}(z; \tau) - 2\pi i \sum_{u \in \text{Sing}_p \varphi(\cdot; \tau)} \text{Res}_{v=u} \left[\tilde{f}_v(z; \tau)\varphi(v; \tau) \right],$$

with h_l as in the theorem. Since $v \mapsto \tilde{f}_v(z; \tau)\varphi(v; \tau)$ is invariant under translation by a lattice point, so is $u \mapsto \text{Res}_{v=u} \left[\tilde{f}_v(z; \tau)\varphi(v; \tau) \right]$. Hence we can replace $\sum_{u \in \text{Sing}_p \varphi(\cdot; \tau)}$ by

$\sum_{u \in \text{Sing} \varphi(\cdot; \tau) \bmod \Lambda_\tau}$. So far we have only proven the identity for $z \in P_p$. However, both sides satisfy (E), so the identity holds for all $z \in \mathbf{C}$, $z \notin \text{Sing} \varphi(\cdot; \tau)$.

In the rest of the proof we assume that φ satisfies (M). Let

$$\tilde{\varphi}(z; \tau) := 2\pi i \sum_{u \in \text{Sing} \varphi(\cdot; \tau) \bmod \Lambda_\tau} \text{Res}_{v=u} \left[\tilde{f}_v(z; \tau)\varphi(v; \tau) \right].$$

From the first part of the theorem we see that $\varphi + \tilde{\varphi}$ is a holomorphic function, which satisfies (E), and

$$\varphi(z; \tau) + \tilde{\varphi}(z; \tau) = \sum_{l \bmod 2m} h_l(\tau)\vartheta_{m,l}(z; \tau)$$

If we can show that $\tilde{\varphi}$ also satisfies (M), then the second part of the theorem follows from the second part of Theorem 3.1 applied to $\varphi + \tilde{\varphi}$.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbf{Z})$. If u is a pole of $\varphi(\cdot; \gamma\tau)$, then $u' = (c\tau + d)u$ is a pole of $\varphi(\cdot; \tau)$, and $\Lambda_\tau = (c\tau + d)\Lambda_{\gamma\tau}$. Hence

$$\text{Sing} \varphi(\cdot; \gamma\tau) \bmod \Lambda_{\gamma\tau} = (c\tau + d) \text{Sing} \varphi(\cdot; \tau) \bmod \Lambda_\tau.$$

Using (5) of Proposition 3.5 we find

$$\begin{aligned} & \tilde{f}_{\frac{v}{c\tau+d}} \left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d} \right) \left(\frac{v}{c\tau+d} - \frac{u}{c\tau+d} \right)^s \varphi \left(\frac{v}{c\tau+d}; \frac{a\tau+b}{c\tau+d} \right) \\ &= (c\tau+d)^{k+1-s} e^{2\pi i m c z^2 / (c\tau+d)} \tilde{f}_v(z; \tau) (v-u)^s \varphi(v; \tau). \end{aligned}$$

If we apply $\frac{1}{(s-1)!} \frac{\partial^{s-1}}{\partial v^{s-1}} \Big|_{v=u}$ to both sides and multiply by $(c\tau + d)^{s-1}$ we find

$$\begin{aligned} & \operatorname{Res}_{v=\frac{u}{c\tau+d}} \left[\tilde{f}_{\frac{v}{c\tau+d}} \left(\frac{z}{c\tau+d}; \frac{a\tau+d}{c\tau+d} \right) \varphi \left(\frac{v}{c\tau+d}; \frac{a\tau+d}{c\tau+d} \right) \right] \\ &= (c\tau + d)^k e^{2\pi i m c z^2 / (c\tau+d)} \operatorname{Res}_{v=u} \left[\tilde{f}_v(z; \tau) \varphi(v; \tau) \right]. \end{aligned}$$

Hence

$$\begin{aligned} & \tilde{\varphi} \left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d} \right) \\ &= 2\pi i \sum_{u \in \operatorname{Sing} \varphi(\cdot; \gamma\tau) \bmod \Lambda_{\gamma\tau}} \operatorname{Res}_{v=u} \left[\tilde{f}_u \left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d} \right) \varphi \left(\frac{v}{c\tau+d}; \frac{a\tau+d}{c\tau+d} \right) \right] \\ &= 2\pi i \sum_{u' \in \operatorname{Sing} \varphi(\cdot; \tau) \bmod \Lambda_{\tau}} \operatorname{Res}_{v=\frac{u'}{c\tau+d}} \left[\tilde{f}_u \left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d} \right) \varphi \left(\frac{v}{c\tau+d}; \frac{a\tau+d}{c\tau+d} \right) \right] \\ &= 2\pi i \sum_{u' \in \operatorname{Sing} \varphi(\cdot; \tau) \bmod \Lambda_{\tau}} (c\tau + d)^k e^{2\pi i m c z^2 / (c\tau+d)} \operatorname{Res}_{v=u'} \left[\tilde{f}_v(z; \tau) \varphi(v; \tau) \right] \\ &= (c\tau + d)^k e^{2\pi i m c z^2 / (c\tau+d)} \tilde{\varphi}(z; \tau), \end{aligned}$$

where in the second step we have substituted $u = \frac{u'}{c\tau+d}$. \square

3.4 Simple poles

If all the poles of φ are simple, the theorem from the previous section reduces to

Corollary 3.10 *Let φ be such that $z \mapsto \varphi(z; \tau)$, for fixed $\tau \in \mathcal{H}$, is a meromorphic function having only simple poles. If φ satisfies (E), then*

$$\varphi(z; \tau) = \sum_{l \bmod 2m} h_l(\tau) \vartheta_{m,l}(z; \tau) + \sum_{u \in \operatorname{Sing} \varphi(\cdot; \tau) \bmod \Lambda_{\tau}} d_u(\tau) \tilde{f}_u(z; \tau),$$

with

$$\begin{aligned} \operatorname{Sing} \varphi(\cdot; \tau) &:= \{u \in \mathbf{C} \mid z \mapsto \varphi(z; \tau) \text{ has a pole in } u\}, \\ \Lambda_{\tau} &= \mathbf{Z}\tau + \mathbf{Z}, \\ d_u(\tau) &= -2\pi i \operatorname{Res}_{z=u} \varphi(z; \tau), \end{aligned}$$

and, for $0 \leq l \leq 2m-1$ and any $p \in \mathbf{C}$ such that there are no poles on the boundary ∂P_p of $P_p := p + (0, 1)\tau + (0, 1)$,

$$h_l(\tau) = e^{-\pi i l^2 \tau / 2m} \int_p^{p+1} \varphi(z; \tau) e^{-2\pi i l z} dz + \frac{1}{2} \sum_{u \in \operatorname{sing}_p \varphi(\cdot; \tau)} d_u(\tau) R_{m,l}(u; \tau),$$

with $\text{sing}_p \varphi(\cdot; \tau) = \text{Sing} \varphi(\cdot; \tau) \cap P_p$.

If φ also satisfies (M), then the vector $(h_l)_{l \bmod 2m}$ transforms under the action of $\text{SL}_2(\mathbf{Z})$ as in equations (3.3) and (3.4).

In a special case, the residue function $\tau \mapsto d_u(\tau)$ has modular transformation properties:

Proposition 3.11 *Let φ and $d_u(\tau)$ be as in Corollary 3.10 and suppose that the pole u of $\varphi(\cdot; \tau)$ is of the form $u = \alpha\tau + \beta$, with $\alpha, \beta \in \mathbf{Q}$ independent of τ . Then $\tau \mapsto e^{2\pi i m \alpha^2 \tau} d_u(\tau)$ transforms as a modular form of weight $k - 1$ on some subgroup $\Gamma_{\alpha, \beta}$ of $\text{SL}_2(\mathbf{Z})$.*

Proof: From the definition of d_u we can easily verify that

$$d_{u+\lambda\tau+\mu}(\tau) = e^{-2\pi i m(\lambda^2\tau+2\lambda u)} d_u(\tau),$$

and

$$d_{\frac{u}{c\tau+d}}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{k-1} e^{2\pi i m c u^2 / (c\tau+d)} d_u(\tau).$$

Hence $(u, \tau) \mapsto d_u(\tau)$ transforms as a Jacobi form of weight $k - 1$ and index m . By Theorem 1.3 of [9, pp. 10] we get the desired result. Actually, that theorem also assumes a growth condition, but one can check in the proof that the growth condition is not needed to prove the modular transformation properties. \square

3.5 An example

Define φ by:

$$\begin{aligned} \varphi(z; \tau) &:= \frac{\left(\vartheta_{0,0}(z; \tau) \vartheta_{0, \frac{1}{2}}(z; \tau) \vartheta_{\frac{1}{2}, 0}(z; \tau)\right)^9}{\Delta(\tau) \vartheta_{\frac{1}{2}, \frac{1}{2}}(z; \tau)} \\ &= -i \left(\zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}}\right)^9 \left\{ \frac{1}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} - \left(9\zeta^{\frac{3}{2}} - \zeta^{\frac{1}{2}} + \zeta^{-\frac{1}{2}} - 9\zeta^{-\frac{3}{2}}\right)q + \dots \right\}, \end{aligned}$$

with $\vartheta_{a,b}(z; \tau) := \sum_{\lambda \in a + \mathbf{Z}} e^{\pi i \lambda^2 \tau + 2\pi i \lambda(z+b)}$, $\zeta = e^{2\pi i z}$ and $q = e^{2\pi i \tau}$.

Using Table V on page 36 of [19] we see that φ transforms like a Jacobi form of weight $k = 1$ and index $m = 13$ on the full modular group. (Note that Mumford uses the notation ϑ_{01} , ϑ_{10} and ϑ_{11} for the functions denoted here by $\vartheta_{0, \frac{1}{2}}$, $\vartheta_{\frac{1}{2}, 0}$ and $\vartheta_{\frac{1}{2}, \frac{1}{2}}$. Also there's a mistake in the 4th formula on the right: it should read $\vartheta_{11}(z/\tau, -1/\tau) = -i(-i\tau)^{\frac{1}{2}} \exp(\pi i z^2/\tau) \vartheta_{11}(z, \tau)$.)

The function φ is meromorphic in z with simple poles in $\mathbf{Z}\tau + \mathbf{Z}$. If we take $p = -\frac{1}{2}\tau - \frac{1}{2}$ then $\text{sing}_p \varphi(\cdot; \tau) = \{0\}$. Further

$$\begin{aligned} \text{Res}_{z=0} \varphi(z; \tau) &= \frac{\left(\vartheta_{0,0}(0; \tau)\vartheta_{0,\frac{1}{2}}(0; \tau)\vartheta_{\frac{1}{2},0}(0; \tau)\right)^9}{\Delta(\tau)\vartheta'_{\frac{1}{2},\frac{1}{2}}(0; \tau)} = -\frac{1}{\pi^9} \frac{\vartheta'_{\frac{1}{2},\frac{1}{2}}(0; \tau)^8}{\Delta(\tau)} \\ &= -\frac{1}{\pi^9} \frac{(-2\pi\eta(\tau)^3)^8}{\Delta(\tau)} = -\frac{128}{\pi}, \end{aligned}$$

where we have used Jacobi's derivative formula (see [19, pp. 64]), and (10) of Proposition 1.3.

Corollary 3.10 gives:

$$\varphi(z; \tau) = \sum_{l \bmod 26} h_l(\tau)\vartheta_{13,l}(z; \tau) + 512i\tilde{f}_0(z; \tau) \quad (3.8)$$

with

$$h_l(\tau) = e^{-\pi i l^2 \tau / 26} \int_p^{p+1} \varphi(z; \tau) e^{-2\pi i l z} dz + 256i R_{13,l}(0; \tau), \quad (3.9)$$

for $0 \leq l \leq 25$. According to the corollary the h_l transform as a vector-valued modular form of weight $\frac{1}{2}$.

Since φ is holomorphic as a function of τ , the Fourier coefficients

$$e^{-\pi i l^2 \tau / 26} \int_p^{p+1} \varphi(z; \tau) e^{-2\pi i l z} dz$$

are holomorphic as a function of τ . In particular they are eigenfunctions of $\Omega_{\frac{1}{2}}$ with eigenvalue $\frac{3}{16}$. Using (2) of Proposition 3.7 we see that $\tau \mapsto R_{13,l}(0; \tau)$ is also an eigenfunction of $\Omega_{\frac{1}{2}}$ with eigenvalue $\frac{3}{16}$. Hence

$$\Omega_{\frac{1}{2}} h_l = \frac{3}{16} h_l.$$

So the h_l form a vector-valued real-analytic modular form of weight $\frac{1}{2}$. The transformations are:

$$h_l(\tau + 1) = e^{-\pi i l^2 / 26} h_l(\tau)$$

and

$$h_l\left(-\frac{1}{\tau}\right) = i\sqrt{-i\tau} \frac{1}{\sqrt{26}} \sum_{\nu \bmod 26} e^{\pi i l \nu / 13} h_\nu(\tau)$$

Using (1) of Proposition 3.7 we see

$$\frac{\partial}{\partial \bar{\tau}} h_l(\tau) = 256\sqrt{13} y^{-\frac{1}{2}} \sum_{\lambda \in \frac{1}{26} + \mathbf{Z}} \lambda e^{-26\pi i \lambda^2 \bar{\tau}}.$$

Note that $\sum_{\lambda \in \frac{1}{26} + \mathbf{Z}} \lambda e^{-26\pi i \lambda^2 \tau}$ is the complex conjugate of $\sum_{\lambda \in \frac{1}{26} + \mathbf{Z}} \lambda e^{26\pi i \lambda^2 \tau}$, which is a theta function of weight $3/2$.

Summarizing, we have proved:

Proposition 3.12 *Let φ be the function given by*

$$\varphi(z; \tau) = \frac{\left(\vartheta_{0,0}(z; \tau) \vartheta_{0, \frac{1}{2}}(z; \tau) \vartheta_{\frac{1}{2}, 0}(z; \tau) \right)^9}{\Delta(\tau) \vartheta_{\frac{1}{2}, \frac{1}{2}}(z; \tau)}.$$

Then φ can be decomposed as in (3.8), with h_l as in (3.9), and $(h_l)_{l \bmod 26}$ is a vector-valued real-analytic modular form of weight $1/2$, with eigenvalue $3/16$ for the weight $1/2$ Casimir operator.

This is a very special example: It has been constructed in such a way that the d_u are constant (as a function of τ). As a result the h_l are eigenfunctions of a Casimir operator. However, in general the d_u will not be constant and the h_l will not be eigenfunctions of a Casimir operator. So in general we do not end up with a real-analytic modular form.