Chapter 4

Mock ϑ-functions

4.1 Introduction

Mock ϑ-functions were introduced by S. Ramanujan in the last letter he wrote to G.H. Hardy, dated January, 1920. For a photocopy of the mathematical part of this letter see [21, pp. 127–131] (also reproduced in [4]). In this letter, Ramanujan provided a list of 17 mock ϑ-functions, together with identities they satisfy. Ramanujan divided his list of functions into "third order", "fifth order" and "seventh order" functions, but did not say what he meant. There's still no formal definition of "order", but known identities for these mock ϑ-functions make it clear that they are related to the numbers 3, 5 and 7. Therefore we regard the order of a mock ϑ-function merely as a convenient label, which may or may not have a deeper significance.

In his letter, Ramanujan explained what he meant by a mock ϑ-function. In [5] we find a formal definition. Slightly rephrased it reads: a mock ϑ-function is a function f of the complex variable q, defined by a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

(1) infinitely many roots of unity are exponential singularities,

(2) for every root of unity ξ there is a ϑ-function $\vartheta_\xi(q)$ such that the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially (presumably with only finitely many of the ϑ_ξ being different),

(3) there is no ϑ-function that works for all ξ, i.e. f is not the sum of two functions, one of which is a ϑ-function and the other a function which is bounded in all roots of unity.

(When Ramanujan refers to ϑ-functions, he means sums, products, and quotients of series of the form $\sum_{n\in\mathbf{Z}} \epsilon^n q^{an^2+bn}$ with $a, b \in \mathbf{Q}$ and $\epsilon = -1, 1$).

The 17 functions given by Ramanujan indeed satisfy conditions (1) and (2) (see [26], [27] and [23]). However no proof has ever been given that they also satisfy condition (3). Watson (see [26]) proved a very weak form of condition (3) for the "third order" mock ϑ-functions, namely, that they are not equal to ϑ-functions.

In this chapter we will see that condition (3) is not satisfied if we strengthen it slightly. Indeed, we shall discuss vector-valued mock ϑ-functions F for which there is a vector-valued real-analytic modular form H such that $F - H$ is bounded in all roots of unity.

There are several ways to get these results: For example let us consider the "fifth order" mock ϑ-function (using Watson's notation)

$$f_0(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n},$$

with $(a)_n = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1})$. Andrews (see [2]) showed that

$$f_0(q) = \frac{1}{(q)_\infty} \sum_{n\geq 0} \sum_{|j|\leq n} (-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}(1-q^{4n+2}), \tag{4.1}$$

with $(q)_\infty = \prod_{n=1}^{\infty}(1-q^n)$. Using this identity, Andrews (see [3]) showed that $f_0(q)$ can be seen as a Fourier coefficient of a certain quotient of Jacobi theta functions. Next Hickerson (see [13]) showed that $f_0(q)$ can be related to a sum similar to the Lerch sum discussed in Chapter I. Similar results have been found for most other mock ϑ-functions. Hence we can extend the results of Chapter I to include these "Lerch-like" sums and thereby get the transformation properties of f_0. We may also use the techniques from Chapter 3 and the representation of the mock ϑ-functions as a Fourier coefficient of a meromorphic Jacobi form, to find the transformation properties. However, we will use (4.1) and similar identities for the other mock ϑ-functions and apply the results from Chapter 2.

4.2 General results

Before we start with the mock ϑ-functions, we derive some general results, which we will use repeatedly.

Definition 4.1 For $a, b \in \mathbf{R}$ and $\tau \in \mathcal{H}$ we define

$$R_{a,b}(\tau) = \sum_{\nu\in a+\mathbf{Z}} \operatorname{sgn}(\nu)\beta(2\nu^2 y)\ e^{-\pi i\nu^2\tau-2\pi i\nu b},$$

with $y = \operatorname{Im}(\tau)$ and β as in Lemma 1.7.

This function, a kind of non-holomorphic unary theta-series, is a slight modification of the function R studied in Chapter 1 (cf. (1) of the following proposition) and is also similar to the function $R_{m,l}(z;\tau)$ defined in Definition 3.4.

Proposition 4.2 *Let $a \in (0,1)$, $b \in \mathbf{R}$ and $\tau \in \mathcal{H}$. We have*

(1) $R_{a,b}(\tau) = ie^{-\pi i(a-\frac{1}{2})^2\tau - 2\pi i(a-\frac{1}{2})b}\ R\Big((a-\frac{1}{2})\tau + b + \frac{1}{2};\tau\Big)$*, with R as defined in Lemma 1.8.*

(2) $R_{a,b}(\tau) = -i\displaystyle\int_{-\overline{\tau}}^{i\infty} \frac{g_{a,-b}(z)}{\sqrt{-i(z+\tau)}}\,dz$*, with $g_{a,b}$ as in Definition 1.14.*

(3) If $\xi \in \mathbf{Q}$, then $R_{a,b}(\tau)$ is bounded as $\tau \downarrow \xi$.

(4) $\frac{\partial}{\partial\overline{\tau}} R_{a,b}(\tau) = -i\frac{1}{\sqrt{2y}}\ g_{a,-b}(-\overline{\tau})$.

(5) $\tau \mapsto R_{a,b}(\tau)$ is an eigenfunction of the weight 1/2 Casimir operator $\Omega_{\frac{1}{2}} = -4y^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}} + iy\frac{\partial}{\partial\overline{\tau}} + \frac{3}{16}$, with eigenvalue $\frac{3}{16}$.

Proof: (1) By definition we have

$$\begin{aligned} &R\Big(\Big(a-\frac{1}{2}\Big)\tau + b + \frac{1}{2};\tau\Big) \\ &= \sum_{\nu\in\frac{1}{2}+\mathbf{Z}} \Big\{\mathrm{sgn}(\nu) - E\Big(\Big(\nu + a - \frac{1}{2}\Big)\sqrt{2y}\Big)\Big\}\ (-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu((a-\frac{1}{2})\tau + b + \frac{1}{2})}. \end{aligned}$$

Using Lemma 1.7, we write $\mathrm{sgn}(\nu) - E\left(\left(\nu + a - \frac{1}{2}\right)\sqrt{2y}\right)$ as the sum of $\mathrm{sgn}(\nu) - \mathrm{sgn}(\nu + a - \frac{1}{2})$ and $\mathrm{sgn}(\nu + a - \frac{1}{2})\beta(2(\nu + a - \frac{1}{2})^2 y)$. We see that $\mathrm{sgn}(\nu) - \mathrm{sgn}(\nu + a - \frac{1}{2}) = 0$ for all $\nu \in \frac{1}{2} + \mathbf{Z}$, since $a \in (0,1)$. Hence

$$\begin{aligned} &R\Big(\Big(a-\frac{1}{2}\Big)\tau + b + \frac{1}{2};\tau\Big) = \\ &\sum_{\nu\in\frac{1}{2}+\mathbf{Z}} \mathrm{sgn}\Big(\nu + a - \frac{1}{2}\Big)\beta\Big(2\Big(\nu + a - \frac{1}{2}\Big)^2 y\Big)\ (-1)^{\nu-\frac{1}{2}} e^{-\pi i\nu^2\tau - 2\pi i\nu((a-\frac{1}{2})\tau + b + \frac{1}{2})} \\ &= e^{\pi i(a-\frac{1}{2})^2\tau + 2\pi i(a-\frac{1}{2})(b+\frac{1}{2})} \sum_{\nu\in a+\mathbf{Z}} \mathrm{sgn}(\nu)\beta(2\nu^2 y)\ (-1)^{\nu-a} e^{-\pi i\nu^2\tau - 2\pi i\nu(b+\frac{1}{2})} \\ &= -ie^{\pi i(a-\frac{1}{2})^2\tau + 2\pi i(a-\frac{1}{2})b}\ R_{a,b}(\tau), \end{aligned}$$

where we have substituted $\nu \to \nu - a + \frac{1}{2}$ in the second step.
(2) Use (1) of this proposition and (1) of Theorem 1.16 with a replaced by $a - \frac{1}{2}$ and b replaced by $-b - \frac{1}{2}$.
(3) This follows directly from (2) and the fact that $\lim_{z\downarrow\xi} g_{a,-b}(z) = 0$ for all $\xi \in \mathbf{Q}$.
(4) This follows directly from (2) by taking $\frac{\partial}{\partial\overline{\tau}}$ on both sides.
(5) From (4) we see that $\tau \mapsto \sqrt{y}\frac{\partial}{\partial\overline{\tau}} R_{a,b}(\tau)$ is anti-holomorphic, so

$$\frac{\partial}{\partial\tau}\sqrt{y}\frac{\partial}{\partial\overline{\tau}} R_{a,b}(\tau) = 0.$$

We can write the operator $\Omega_{\frac{1}{2}} = -4y^2 \frac{\partial^2}{\partial\tau\partial\overline{\tau}} + iy\frac{\partial}{\partial\overline{\tau}} + \frac{3}{16}$ as

$$\Omega_{\frac{1}{2}} = \frac{3}{16} - 4y^{3/2}\frac{\partial}{\partial\tau}\sqrt{y}\frac{\partial}{\partial\overline{\tau}}.$$

Hence $\Omega_{\frac{1}{2}} R_{a,b} = \frac{3}{16} R_{a,b}$. □

We now return to the general setup of Chapter 2, i.e. indefinite ϑ-series for a quadratic form Q of type $(r-1,1)$. In the next proposition, we will rewrite

$$\sum_{\nu\in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)},$$

(this is the same series as in (2.10)) for $c \in C_Q \cap \mathbf{Z}^r$. In order to do so, we write $\nu = \mu + nc$ with $\mu \in a + \mathbf{Z}^r$, $n \in \mathbf{Z}$, such that $\frac{B(c,\mu)}{2Q(c)} \in [0,1)$. Since $c \in \mathbf{Z}^r$ we can write

$$\left\{\mu \in a + \mathbf{Z}^r \,\middle|\, \frac{B(c,\mu)}{2Q(c)} \in [0,1)\right\} = \bigsqcup_{\mu_0\in P_0}\Big(\mu_0 + \langle c\rangle_{\mathbf{Z}}^{\perp}\Big)$$

(disjoint union), for a suitable finite set P_0, with $\langle c\rangle_{\mathbf{Z}}^{\perp} := \{\xi \in \mathbf{Z}^r \mid B(c,\xi) = 0\}$. We can now state the result:

Proposition 4.3 *Let $c \in \mathbf{C}_Q \cap \mathbf{Z}^r$ be primitive. Then there is a finite set P_0 (see above), such that*

$$\sum_{\nu\in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right) e^{2\pi i Q(\nu)\tau + 2\pi i B(\nu,b)}$$
$$= -\sum_{\mu_0\in P_0} R_{\frac{B(c,\mu_0)}{2Q(c)},-B(c,b)}(-2Q(c)\tau) \cdot \sum_{\xi\in\mu_0^{\perp}+\langle c\rangle_{\mathbf{Z}}^{\perp}} e^{2\pi i Q(\xi)\tau + 2\pi i B(\xi,b^{\perp})},$$

with $\mu_0^{\perp} = \mu_0 - \frac{B(c,\mu_0)}{2Q(c)}c$ and $b^{\perp} = b - \frac{B(c,b)}{2Q(c)}c$.

Remark 4.4 Since $\mu_0^{\perp} + \langle c\rangle_{\mathbf{Z}}^{\perp}$ is a shifted $(r-1)$-dimensional lattice, on which Q is positive definite, the inner sum

$$\sum_{\xi\in\mu_0^{\perp}+\langle c\rangle_{\mathbf{Z}}^{\perp}} e^{2\pi i Q(\xi)\tau + 2\pi i B(\xi,b^{\perp})}$$

is a classical (positive definite) theta function, and is in particular modular of weight $(r-1)/2$.

Proof: We write $\nu = \mu_0 + \xi + nc$, with $\mu_0 \in P_0$, $\xi \in \langle c\rangle_{\mathbf{Z}}^{\perp}$ and $n \in \mathbf{Z}$. Set $\mu_0^{\perp} = \mu_0 - \frac{B(c,\mu_0)}{2Q(c)}c$. Then $B(c,\mu_0^{\perp}) = 0$ and

$$\nu = \Big(n + \frac{B(c,\mu_0)}{2Q(c)}\Big)c + \mu_0^{\perp} + \xi,$$

so

$$\sum_{\nu\in a+\mathbf{Z}^r} \operatorname{sgn}\Big(B(c,\nu)\Big)\beta\left(-\frac{B(c,\nu)^2}{Q(c)}y\right)e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,b)}$$

$$= -\sum_{\mu_0\in P_0}\sum_{\xi\in\langle c\rangle^{\perp}_{\mathbf{Z}}}\sum_{n\in\mathbf{Z}}\operatorname{sgn}\Big(n+\frac{B(c,\mu_0)}{2Q(c)}\Big)\beta\left(-4Q(c)\Big(n+\frac{B(c,\mu_0)}{2Q(c)}\Big)^2y\right)\cdot$$

$$\cdot\, e^{2\pi iQ(c)(n+\frac{B(c,\mu_0)}{2Q(c)})^2\tau+2\pi iQ(\mu_0^{\perp}+\xi)\tau+2\pi iB(c,b)(n+\frac{B(c,\mu_0)}{2Q(c)})+2\pi iB(\mu_0^{\perp}+\xi,b)}$$

$$= -\sum_{\mu_0\in P_0}R_{\frac{B(c,\mu_0)}{2Q(c)},-B(c,b)}(-2Q(c)\tau)\cdot\sum_{\alpha\in\mu_0^{\perp}+\langle c\rangle^{\perp}_{\mathbf{Z}}}e^{2\pi iQ(\alpha)\tau+2\pi iB(\alpha,b)}.$$

If we use $B(\alpha,b)=B(\alpha,b^{\perp})$ for all $\alpha\in\mu_0^{\perp}+\langle c\rangle^{\perp}_{\mathbf{Z}}$, we get the desired result. □

4.3 The seventh order mock ϑ-functions

In this section we deal with the "seventh order" mock ϑ-functions from Ramanujan's letter. In [12, pp. 666] we find the following (slightly rewritten) identities:

$$(q)_\infty\mathcal{F}_0(q)=\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)(-1)^{r+s}q^{\frac32r^2+4rs+\frac32s^2+\frac12r+\frac12s}$$

$$(q)_\infty\mathcal{F}_1(q)=\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)(-1)^{r+s}q^{\frac32r^2+4rs+\frac32s^2+\frac52r+\frac52s+1}$$

$$(q)_\infty\mathcal{F}_2(q)=\Big(\sum_{r,s\geq 0}-\sum_{r,s<0}\Big)(-1)^{r+s}q^{\frac32r^2+4rs+\frac32s^2+\frac32r+\frac32s}.$$

We will use these as the definitions of the mock ϑ-functions. We rewrite these identities ($\zeta_n:=e^{2\pi i/n}$):

$$2\eta(\tau)\zeta_{14}\;q^{-\frac{1}{168}}\mathcal{F}_0(q)=\sum_{\nu\in\frac{1}{14}e+\mathbf{Z}^r}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\;e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{1}{14}e)}$$

$$2\eta(\tau)\zeta_{14}\;q^{\frac{47}{168}}\mathcal{F}_2(q)=\sum_{\nu\in\frac{3}{14}e+\mathbf{Z}^r}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\;e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{5}{14}e)}$$

$$2\eta(\tau)\zeta_{14}\;q^{-\frac{25}{168}}\mathcal{F}_1(q)=\sum_{\nu\in\frac{5}{14}e+\mathbf{Z}^r}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\;e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{3}{14}e)},$$

with $A=\left(\begin{smallmatrix}3&4\\4&3\end{smallmatrix}\right)$, $c_1=\left(\begin{smallmatrix}-3\\4\end{smallmatrix}\right)$, $c_2=\left(\begin{smallmatrix}-4\\3\end{smallmatrix}\right)$ and $e:=\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. We have $B(c_1,c_2)=-28$ and $Q(c_1)=Q(c_2)=-\frac{21}{2}$. If we choose C_Q such that $c_1\in C_Q$ then also $c_2\in C_Q$.

We collect these three mock ϑ-functions into a single vector-valued mock ϑ-function

$$F_7(\tau):=\begin{pmatrix}q^{-\frac{1}{168}}\mathcal{F}_0(q)\\ q^{\frac{47}{168}}\mathcal{F}_2(q)\\ q^{-\frac{25}{168}}\mathcal{F}_1(q)\end{pmatrix}.$$

To express its modular transformation behaviour, we also introduce

$$H_7(\tau) := \frac{\zeta_{14}^{-1}}{2\eta(\tau)} \begin{pmatrix} \vartheta_{\frac{1}{14}e,\frac{1}{14}e} \\ \vartheta_{\frac{3}{14}e,\frac{5}{14}e} \\ \vartheta_{\frac{5}{14}e,\frac{3}{14}e} \end{pmatrix} (\tau),$$

with $\vartheta_{a,b}$ as in Definition 2.1, and

$$G_7(\tau) := -\begin{pmatrix} \zeta_{84}^{-13} R_{\frac{13}{42},-\frac{1}{2}} + \zeta_{84} R_{\frac{41}{42},-\frac{1}{2}} \\ \zeta_{84}^{29} R_{\frac{11}{42},-\frac{5}{2}} + \zeta_{84}^{-41} R_{\frac{25}{42},-\frac{5}{2}} \\ \zeta_{28}^{5} R_{\frac{23}{42},-\frac{3}{2}} + \zeta_{28}^{-9} R_{\frac{37}{42},-\frac{3}{2}} \end{pmatrix} (21\tau).$$

Note that the components of H_7 are the quotient of a (real-analytic) binary theta series by η, while the components of G_7 are (real-analytic) unary theta series.

Proposition 4.5 *We have*

$$F_7 = H_7 + G_7,$$

where

(1) *The function H_7 is a (vector-valued) real-analytic modular form of weight 1/2, satisfying*

$$H_7(\tau+1) = \begin{pmatrix} \zeta_{168}^{-1} & 0 & 0 \\ 0 & \zeta_{168}^{47} & 0 \\ 0 & 0 & \zeta_{168}^{-25} \end{pmatrix} H_7(\tau),$$

and

$$H_7\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\frac{2}{\sqrt{7}} \begin{pmatrix} \sin\frac{\pi}{7} & \sin\frac{3\pi}{7} & \sin\frac{2\pi}{7} \\ \sin\frac{3\pi}{7} & -\sin\frac{2\pi}{7} & \sin\frac{\pi}{7} \\ \sin\frac{2\pi}{7} & \sin\frac{\pi}{7} & -\sin\frac{3\pi}{7} \end{pmatrix} H_7(\tau),$$

and is an eigenfunction of the Casimir operator $\Omega_{\frac{1}{2}}$, with eigenvalue $\frac{3}{16}$.

(2) *The function G_7 is bounded if $\tau \downarrow \xi$, with $\xi \in \mathbf{Q}$.*

Proof: We consider the functions $\vartheta_{\frac{1}{14}e,\frac{1}{14}e}$, $\vartheta_{\frac{3}{14}e,\frac{5}{14}e}$ and $\vartheta_{\frac{5}{14}e,\frac{3}{14}e}$. Using (4) and (2) of Corollary 2.9 we see

$$\begin{pmatrix} \vartheta_{\frac{1}{14}e,\frac{1}{14}e} \\ \vartheta_{\frac{3}{14}e,\frac{5}{14}e} \\ \vartheta_{\frac{5}{14}e,\frac{3}{14}e} \end{pmatrix} (\tau+1) = \begin{pmatrix} \zeta_{28} & 0 & 0 \\ 0 & \zeta_{28}^{9} & 0 \\ 0 & 0 & \zeta_{28}^{25} \end{pmatrix} \begin{pmatrix} \vartheta_{\frac{1}{14}e,\frac{1}{14}e} \\ \vartheta_{\frac{3}{14}e,\frac{5}{14}e} \\ \vartheta_{\frac{5}{14}e,\frac{3}{14}e} \end{pmatrix} (\tau).$$

Using Corollary 2.9 and $\vartheta_{s,\frac{1}{14}e} = -e^{2\pi i(s_1+s_2)}\vartheta_{e-s,\frac{1}{14}e}$ for all $s = \left(\begin{smallmatrix} s_1 \\ s_2 \end{smallmatrix}\right) \in \mathbf{R}^2$, which we get from (1), (2) and (3) of Corollary 2.9, we obtain

$$\begin{pmatrix} \vartheta_{\frac{1}{14}e,\frac{1}{14}e} \\ \vartheta_{\frac{3}{14}e,\frac{5}{14}e} \\ \vartheta_{\frac{5}{14}e,\frac{3}{14}e} \end{pmatrix} \left(-\frac{1}{\tau}\right) = -i\tau\frac{2}{\sqrt{7}} \begin{pmatrix} \sin\frac{\pi}{7} & \sin\frac{3\pi}{7} & \sin\frac{2\pi}{7} \\ \sin\frac{3\pi}{7} & -\sin\frac{2\pi}{7} & \sin\frac{\pi}{7} \\ \sin\frac{2\pi}{7} & \sin\frac{\pi}{7} & -\sin\frac{3\pi}{7} \end{pmatrix} \begin{pmatrix} \vartheta_{\frac{1}{14}e,\frac{1}{14}e} \\ \vartheta_{\frac{3}{14}e,\frac{5}{14}e} \\ \vartheta_{\frac{5}{14}e,\frac{3}{14}e} \end{pmatrix} (\tau).$$

If we use $\eta(\tau+1) = \zeta_{24}\eta(\tau)$ and $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\ \eta(\tau)$, we get the transformations for H_7.

Using Proposition 4.3 we see ($P_0 = \left\{-\frac{13}{14}e, -\frac{27}{14}e, -\frac{41}{14}e\right\}$ and $\langle c\rangle^\perp_{\mathbf{Z}} = \{\xi \in \mathbf{Z}^2 \mid \xi_1 = 0\} = \left\{\left(\begin{smallmatrix} 0 \\ \xi_2 \end{smallmatrix}\right) \middle| \xi_2 \in \mathbf{Z}\right\}$)

$$\begin{aligned} &\sum_{\nu\in\frac{1}{14}e+\mathbf{Z}^r} \operatorname{sgn}(B(c_1,\nu))\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{1}{14}e)} \\ &= -R_{\frac{13}{42},-\frac{1}{2}}(21\tau) \sum_{\mu_2\in-\frac{13}{6}+\mathbf{Z}} e^{3\pi i\mu_2^2\tau+\pi i\mu_2} - R_{\frac{27}{42},-\frac{1}{2}}(21\tau) \sum_{\mu_2\in-\frac{27}{6}+\mathbf{Z}} e^{3\pi i\mu_2^2\tau+\pi i\mu_2} \\ &\quad - R_{\frac{41}{42},-\frac{1}{2}}(21\tau) \sum_{\mu_2\in-\frac{41}{6}+\mathbf{Z}} e^{3\pi i\mu_2^2\tau+\pi i\mu_2} \\ &= -\eta(\tau)\left(\zeta_{12}^{-1}R_{\frac{13}{42},-\frac{1}{2}}(21\tau) + \zeta_{12}R_{\frac{41}{42},-\frac{1}{2}}(21\tau)\right). \end{aligned}$$

Similarly we find

$$\begin{aligned} &\sum_{\nu\in\frac{3}{14}e+\mathbf{Z}^r} \operatorname{sgn}(B(c_1,\nu))\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{5}{14}e)} \\ &\qquad = -\eta(\tau)\left(\zeta_{12}^5R_{\frac{11}{42},-\frac{5}{2}}(21\tau) + \zeta_{12}^{-5}R_{\frac{25}{42},-\frac{5}{2}}(21\tau)\right) \\ &\sum_{\nu\in\frac{5}{14}e+\mathbf{Z}^r} \operatorname{sgn}(B(c_1,\nu))\beta\left(-\frac{B(c_1,\nu)^2}{Q(c_1)}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{3}{14}e)} \\ &\qquad = -\eta(\tau)\left(\zeta_4R_{\frac{23}{42},-\frac{3}{2}}(21\tau) + \zeta_4^{-1}R_{\frac{37}{42},-\frac{3}{2}}(21\tau)\right), \end{aligned}$$

and

$$\begin{aligned} &\sum_{\nu\in\frac{1}{14}e+\mathbf{Z}^r} \operatorname{sgn}(B(c_2,\nu))\beta\left(-\frac{B(c_2,\nu)^2}{Q(c_2)}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{1}{14}e)} \\ &\qquad = \eta(\tau)\left(\zeta_{12}^{-1}R_{\frac{13}{42},-\frac{1}{2}}(21\tau) + \zeta_{12}R_{\frac{41}{42},-\frac{1}{2}}(21\tau)\right) \\ &\sum_{\nu\in\frac{3}{14}e+\mathbf{Z}^r} \operatorname{sgn}(B(c_2,\nu))\beta\left(-\frac{B(c_2,\nu)^2}{Q(c_2)}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{5}{14}e)} \\ &\qquad = \eta(\tau)\left(\zeta_{12}^5R_{\frac{11}{42},-\frac{5}{2}}(21\tau) + \zeta_{12}^{-5}R_{\frac{25}{42},-\frac{5}{2}}(21\tau)\right) \end{aligned}$$

$$\sum_{\nu\in\frac{5}{14}e+\mathbf{Z}^r} \operatorname{sgn}(B(c_2,\nu))\beta\left(-\frac{B(c_2,\nu)^2}{Q(c_2)}y\right) e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{3}{14}e)}$$
$$= \eta(\tau)\left(\zeta_4 R_{\frac{23}{42},-\frac{3}{2}}(21\tau)+\zeta_4^{-1}R_{\frac{37}{42},-\frac{3}{2}}(21\tau)\right).$$

Hence, if we write $\rho(\nu;\tau)$ as the sum of the three expressions (2.7), (2.8) and (2.9), we find

$$\begin{aligned}\vartheta_{\frac{1}{14}e,\frac{1}{14}e}(\tau) &= \sum_{\nu\in\frac{1}{14}e+\mathbf{Z}^r}\left\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\right\} e^{2\pi iQ(\nu)\tau+2\pi iB(\nu,\frac{1}{14}e)}\\ &\quad + 2\eta(\tau)\left(\zeta_{12}^{-1}R_{\frac{13}{42},-\frac{1}{2}}(21\tau)+\zeta_{12}R_{\frac{41}{42},-\frac{1}{2}}(21\tau)\right)\\ &= 2\zeta_{14}\, q^{-\frac{1}{168}}\eta(\tau)\mathcal{F}_0(q)+2\eta(\tau)\left(\zeta_{12}^{-1}R_{\frac{13}{42},-\frac{1}{2}}(21\tau)+\zeta_{12}R_{\frac{41}{42},-\frac{1}{2}}(21\tau)\right),\end{aligned}$$

so

$$\frac{\zeta_{14}^{-1}}{2\eta(\tau)}\,\vartheta_{\frac{1}{14}e,\frac{1}{14}e}(\tau) = q^{-\frac{1}{168}}\mathcal{F}_0(q)+\zeta_{84}^{-13}R_{\frac{13}{42},-\frac{1}{2}}(21\tau)+\zeta_{84}R_{\frac{41}{42},-\frac{1}{2}}(21\tau).$$

Similarly, we find

$$\frac{\zeta_{14}^{-1}}{2\eta(\tau)}\,\vartheta_{\frac{3}{14}e,\frac{5}{14}e}(\tau) = q^{\frac{47}{168}}\mathcal{F}_2(q)+\zeta_{84}^{29}R_{\frac{11}{42},-\frac{5}{2}}(21\tau)+\zeta_{84}^{-41}R_{\frac{25}{42},-\frac{5}{2}}(21\tau),$$
$$\frac{\zeta_{14}^{-1}}{2\eta(\tau)}\,\vartheta_{\frac{5}{14}e,\frac{3}{14}e}(\tau) = q^{-\frac{25}{168}}\mathcal{F}_1(q)+\zeta_{28}^{5}R_{\frac{23}{42},-\frac{3}{2}}(21\tau)+\zeta_{28}^{-9}R_{\frac{37}{42},-\frac{3}{2}}(21\tau).$$

So

$$H_7 = F_7 - G_7.$$

Using (5) of Proposition 4.2 and the fact that F_7 is holomorphic, we find

$$\Omega_{\frac{1}{2}}G_7 = \frac{3}{16}G_7$$
$$\Omega_{\frac{1}{2}}F_7 = \frac{3}{16}F_7,$$

and so

$$\Omega_{\frac{1}{2}}H_7 = \frac{3}{16}H_7.$$

Hence H_7 is a vector valued real-analytic modular form of weight 1/2. From (3) of Proposition 4.2 we obtain that G_7 is bounded if $\tau\downarrow\xi$, with $\xi\in\mathbf{Q}$. □

As a corollary we get the description of the non-modularity of the "seventh order" mock ϑ-function F_7. To state the corollary we need the following vector of theta functions of weight 3/2:

$$g_7(\tau) := \begin{pmatrix}\zeta_{84}^{-13}\ g_{\frac{13}{42},\frac{1}{2}}+\zeta_{84}\ g_{\frac{41}{42},\frac{1}{2}}\\ \zeta_{84}^{73}\ g_{\frac{11}{42},\frac{1}{2}}+\zeta_{84}^{59}\ g_{\frac{25}{42},\frac{1}{2}}\\ \zeta_{84}^{61}\ g_{\frac{23}{42},\frac{1}{2}}+\zeta_{84}^{47}\ g_{\frac{37}{42},\frac{1}{2}}\end{pmatrix}(21\tau).$$

This function has the following modular transformation property, which can be verified using standard methods:

$$g_7\Big(-\frac{1}{\tau}\Big) = -M_7(-i\tau)^{3/2}g_7(\tau),$$

with

$$M_7 := \frac{2}{\sqrt{7}}\begin{pmatrix} \sin\frac{\pi}{7} & \sin\frac{3\pi}{7} & \sin\frac{2\pi}{7} \\ \sin\frac{3\pi}{7} & -\sin\frac{2\pi}{7} & \sin\frac{\pi}{7} \\ \sin\frac{2\pi}{7} & \sin\frac{\pi}{7} & -\sin\frac{3\pi}{7} \end{pmatrix}.$$

Corollary 4.6 *We have*

$$F_7(\tau) - \frac{1}{\sqrt{-i\tau}}M_7F_7\Big(-\frac{1}{\tau}\Big) = i\sqrt{21}\int_0^{i\infty}\frac{g_7(z)}{\sqrt{-i(z+\tau)}}\,dz,$$

where we have to integrate each component of the vector, as well as the obvious equation

$$F_7(\tau+1) = \begin{pmatrix} \zeta_{168}^{-1} & 0 & 0 \\ 0 & \zeta_{168}^{47} & 0 \\ 0 & 0 & \zeta_{168}^{-25} \end{pmatrix} F_7(\tau).$$

Proof: According to Proposition 4.5 we have

$$H_7\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}M_7H_7(\tau).$$

If we replace τ by $-1/\tau$ in the equation (or multiply both sides by $M_7/\sqrt{-i\tau}$ and use $M_7^2 = I$) we find

$$H_7(\tau) = \frac{1}{\sqrt{-i\tau}}M_7H_7\Big(-\frac{1}{\tau}\Big),$$

so

$$F_7(\tau) - \frac{1}{\sqrt{-i\tau}}M_7F_7\Big(-\frac{1}{\tau}\Big) = G_7(\tau) - \frac{1}{\sqrt{-i\tau}}M_7G_7\Big(-\frac{1}{\tau}\Big). \tag{4.2}$$

Using (2) of Proposition 4.2 and (2) of Proposition 1.15 we see:

$$G_7(\tau) = i\sqrt{21}\int_{-\overline{\tau}}^{i\infty}\frac{g_7(z)}{\sqrt{-i(z+\tau)}}\,dz. \tag{4.3}$$

Hence

$$\begin{aligned}\frac{1}{\sqrt{-i\tau}}G_7\Big(-\frac{1}{\tau}\Big) &= \frac{i\sqrt{21}}{\sqrt{-i\tau}}\int_{1/\overline{\tau}}^{i\infty}\frac{g_7(z)}{\sqrt{-i(z-1/\tau)}}\,dz \\ &= i\sqrt{21}\int_0^{-\overline{\tau}}\frac{g_7(-1/z)}{\sqrt{1+\tau/z}}\frac{dz}{(-iz)^2},\end{aligned}$$

where we have replaced z by $-1/z$ in the integral. Using the transformation property of g_7, we find

$$\frac{1}{\sqrt{-i\tau}}G_7\Big(-\frac{1}{\tau}\Big) = -i\sqrt{21}M_7\int_0^{-\overline{\tau}} \frac{g_7(z)}{\sqrt{-i(z+\tau)}}\,dz. \tag{4.4}$$

Putting (4.3) and (4.4) in (4.2) we get the desired result. □

Remark 4.7 Using (2) of Theorem 1.16 we could give the non-modularity of F_7 in terms of the function h from Chapter 1. The result is similar to results found by Watson in [26] for the "third order" mock ϑ-functions.

4.4 The fifth order mock ϑ-functions

In this section we deal with eight of the ten "fifth order" mock ϑ-functions from Ramanujan's letter. The remaining two will be discussed in the next section. In [2] we find the following identities:

$$f_0(q) = \frac{1}{(q)_\infty}\sum_{\substack{n=0\\|j|\le n}}^{\infty}(-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2}(1-q^{4n+2}),$$

$$F_0(q) = \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{5n^2+2n-\frac{1}{2}j^2-\frac{1}{2}j}(1+q^{6n+3}),$$

$$1+2\psi_0(q) = \frac{(-q)_\infty}{(q)_\infty}\Big(1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2+n} - 2\sum_{\substack{n=1\\|j|<n}}^{\infty}(-1)^j q^{\frac{5}{2}n^2-\frac{1}{2}n-\frac{3}{2}j^2-\frac{1}{2}j}(1-q^n)\Big),$$

$$\varphi_0(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{\substack{n=0\\|j|\le n}}^{\infty}(-1)^j q^{5n^2+2n-3j^2-j}(1-q^{6n+3}),$$

$$f_1(q) = \frac{1}{(q)_\infty}\sum_{\substack{n=0\\|j|\le n}}^{\infty}(-1)^j q^{\frac{5}{2}n^2+\frac{3}{2}n-j^2}(1-q^{2n+1}),$$

$$F_1(q) = \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{5n^2+4n-\frac{1}{2}j^2-\frac{1}{2}j}(1+q^{2n+1}),$$

$$\psi_1(q) = \frac{(-q)_\infty}{(q)_\infty}\sum_{\substack{n=0\\|j|\le n}}^{\infty}(-1)^j q^{\frac{5}{2}n^2+\frac{3}{2}n-\frac{3}{2}j^2-\frac{1}{2}j}(1-q^{2n+1}),$$

$$\varphi_1(q) = q\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{\substack{n=0\\|j|\le n}}^{\infty}(-1)^j q^{5n^2+4n-3j^2-j}(1-q^{2n+1}).$$

Note that there are mistakes in the 3rd and 8th formula in [2]. We will use these identities as the definitions of the mock ϑ-functions. We write four of these identities in a more suitable form (the other four will be discussed later):

Lemma 4.8 *We have*

$$\begin{aligned}
&2\eta(\tau)q^{-\frac{1}{60}}f_0(q)\\
&\quad=\sum_{\nu\in\left(\begin{smallmatrix}1/10\\0\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)\right)}\\
&2\eta(\tau)q^{\frac{11}{60}}f_1(q)\\
&\quad=\sum_{\nu\in\left(\begin{smallmatrix}3/10\\0\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)\right)}\\
&2\eta(\tau)q^{-\frac{1}{240}}(-1+F_0(q^{\frac12}))\\
&\quad=\sum_{\nu\in\left(\begin{smallmatrix}1/5\\1/4\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}1/2\\1\end{smallmatrix}\right)\right)}\\
&2\eta(\tau)q^{\frac{71}{240}}F_1(q^{\frac12})\\
&\quad=\sum_{\nu\in\left(\begin{smallmatrix}2/5\\1/4\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}1/2\\2\end{smallmatrix}\right)\right)}\\
&2\eta(\tau)\zeta_8^{-1}q^{-\frac{1}{240}}(-1+F_0(-q^{\frac12}))\\
&\quad=\sum_{\nu\in\left(\begin{smallmatrix}1/5\\1/4\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)\right)}\\
&2\eta(\tau)\zeta_8^{-1}q^{\frac{71}{240}}F_1(-q^{\frac12})\\
&\quad=\sum_{\nu\in\left(\begin{smallmatrix}2/5\\1/4\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)\right)},
\end{aligned}$$

with $A=\left(\begin{smallmatrix}5&0\\0&-2\end{smallmatrix}\right)$, $c_1=\left(\begin{smallmatrix}2\\5\end{smallmatrix}\right)$ *and* $c_2=\left(\begin{smallmatrix}-2\\5\end{smallmatrix}\right)$.

Remark 4.9 We have $B(c_1,c_2)=-70$ and $Q(c_1)=Q(c_2)=-15$. If we choose C_Q such that $c_1\in C_Q$ then also $c_2\in C_Q$.

Proof: We have

$$\sum_{\substack{n=0\\|j|\le n}}^{\infty}(-1)^j q^{\frac52 n^2+\frac12 n-j^2}(1-q^{4n+2})$$

$$= \sum_{\substack{n=0\\|j|\le n}}^{\infty} (-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2} - \sum_{\substack{n=0\\|j|\le n}}^{\infty} (-1)^j q^{\frac{5}{2}n^2+\frac{9}{2}n+2-j^2}$$
$$= \Big(\sum_{n+j\ge 0, n-j\ge 0} - \sum_{n+j<0, n-j<0}\Big)(-1)^j q^{\frac{5}{2}n^2+\frac{1}{2}n-j^2},$$

where we have replaced n by $-n-1$ in the second sum. From this we get the first identity. The proof of the 2rd identity is similar.

Using

$$\sum_{j=0}^{2n} q^{-\frac{1}{2}j^2-\frac{1}{2}j} = \sum_{j=-n}^{n} q^{-2j^2-j}, \tag{4.5}$$

we see

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{5n^2+2n-\frac{1}{2}j^2-\frac{1}{2}j}(1+q^{6n+3})$$
$$= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{5n^2+2n-2j^2-j}(1+q^{6n+3})$$
$$= \Big(\sum_{n+j\ge 0, n-j\ge 0} - \sum_{n+j<0, n-j<0}\Big)(-1)^n q^{5n^2+2n-2j^2-j}$$
$$= (q^2;q^2)_\infty + \Big(\sum_{n+j\ge 0, n-j>0} - \sum_{n+j<0, n-j\le 0}\Big)(-1)^n q^{5n^2+2n-2j^2-j}.$$

From this we get the 3rd and 5th identity. Again using (4.5) we see

$$\sum_{n=0}^{\infty}\sum_{j=0}^{2n}(-1)^n q^{5n^2+4n-\frac{1}{2}j^2-\frac{1}{2}j}(1+q^{2n+1})$$
$$= \sum_{n=0}^{\infty}\sum_{j=-n}^{n}(-1)^n q^{5n^2+4n-2j^2-j}(1+q^{2n+1})$$
$$= \Big(\sum_{n+j\ge 0, n-j\ge 0} - \sum_{n+j<0, n-j<0}\Big)(-1)^n q^{5n^2+4n-2j^2-j}.$$

From this we get the 4th and 6th identity. □

We collect these six mock ϑ-functions into a single vector-valued mock ϑ-function

$$F_{5,1}(\tau) := \begin{pmatrix} q^{-\frac{1}{60}} f_0(q) \\ q^{\frac{11}{60}} f_1(q) \\ q^{-\frac{1}{240}}(-1+F_0(q^{\frac{1}{2}})) \\ q^{\frac{71}{240}} F_1(q^{\frac{1}{2}}) \\ q^{-\frac{1}{240}}(-1+F_0(-q^{\frac{1}{2}})) \\ q^{\frac{71}{240}} F_1(-q^{\frac{1}{2}})) \end{pmatrix}.$$

To express its modular transformation behaviour, we also introduce

$$H_{5,1}(\tau) := \frac{1}{2\eta(\tau)} \begin{pmatrix} \vartheta_{\binom{1/10}{0},\binom{0}{1/4}} \\ \vartheta_{\binom{3/10}{0},\binom{0}{1/4}} \\ \vartheta_{\binom{1/5}{1/4},\binom{1/2}{1}} \\ \vartheta_{\binom{2/5}{1/4},\binom{1/2}{2}} \\ \zeta_8\vartheta_{\binom{1/5}{1/4},\binom{0}{1/4}} \\ \zeta_8\vartheta_{\binom{2/5}{1/4},\binom{0}{1/4}} \end{pmatrix}(\tau),$$

with A, c_1 and c_2 as in Lemma 4.8, and

$$G_{5,1}(\tau) := \frac{1}{2} \begin{pmatrix} 2\zeta_{12}R_{\frac{1}{30},\frac{5}{2}} + 2\zeta_{12}^{-1}R_{\frac{11}{30},\frac{5}{2}} \\ 2\zeta_{12}R_{\frac{13}{30},\frac{5}{2}} + 2\zeta_{12}^{-1}R_{\frac{23}{30},\frac{5}{2}} \\ -R_{\frac{19}{60},0} - R_{\frac{29}{60},0} + R_{\frac{49}{60},0} + R_{\frac{59}{60},0} \\ -R_{\frac{13}{60},0} - R_{\frac{23}{60},0} + R_{\frac{43}{60},0} + R_{\frac{53}{60},0} \\ \zeta_{24}^{-5}R_{\frac{19}{60},\frac{5}{2}} + \zeta_{24}^{5}R_{\frac{29}{60},\frac{5}{2}} + \zeta_{24}R_{\frac{49}{60},\frac{5}{2}} + \zeta_{24}^{-1}R_{\frac{59}{60},\frac{5}{2}} \\ \zeta_{24}R_{\frac{13}{60},\frac{5}{2}} + \zeta_{24}^{-1}R_{\frac{23}{60},\frac{5}{2}} + \zeta_{24}^{-5}R_{\frac{43}{60},\frac{5}{2}} + \zeta_{24}^{5}R_{\frac{53}{60},\frac{5}{2}} \end{pmatrix}(30\tau).$$

Proposition 4.10 *We have*

$$F_{5,1} = H_{5,1} + G_{5,1},$$

where

(1) *The function $H_{5,1}$ is a (vector-valued) real-analytic modular form of weight 1/2, satisfying*

$$H_{5,1}(\tau+1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{60}^{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{240}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{240}^{71} \\ 0 & 0 & \zeta_{240}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{240}^{71} & 0 & 0 \end{pmatrix} H_{5,1}(\tau),$$

and

$$H_{5,1}\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\frac{2}{\sqrt{5}}\ M_5\ H_{5,1}(\tau),$$

with

$$M_5 = \begin{pmatrix} 0 & 0 & \sqrt{2}\sin\frac{\pi}{5} & \sqrt{2}\sin\frac{2\pi}{5} & 0 & 0 \\ 0 & 0 & \sqrt{2}\sin\frac{2\pi}{5} & -\sqrt{2}\sin\frac{\pi}{5} & 0 & 0 \\ \frac{1}{\sqrt{2}}\sin\frac{\pi}{5} & \frac{1}{\sqrt{2}}\sin\frac{2\pi}{5} & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}\sin\frac{2\pi}{5} & -\frac{1}{\sqrt{2}}\sin\frac{\pi}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ 0 & 0 & 0 & 0 & \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix},$$

and is an eigenfunction of the Casimir operator $\Omega_{\frac{1}{2}}$, with eigenvalue $\frac{3}{16}$.

(2) *The function $G_{5,1}$ is bounded if $\tau \downarrow \xi$, with $\xi \in \mathbf{Q}$.*

Proof: We consider the function

$$\Theta(\tau) = \begin{pmatrix} \vartheta_{\left(\begin{smallmatrix}1/10\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix}3/10\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix}1/5\\1/4\end{smallmatrix}\right),\left(\begin{smallmatrix}1/2\\1\end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix}2/5\\1/4\end{smallmatrix}\right),\left(\begin{smallmatrix}1/2\\2\end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix}1/5\\1/4\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix}2/5\\1/4\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)} \end{pmatrix}(\tau).$$

Using (4) and (2) of Corollary 2.9 we see

$$\Theta(\tau+1) = \begin{pmatrix} \zeta_{40} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{40}^{9} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{80}^{13} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{80}^{37} \\ 0 & 0 & \zeta_{80}^{-7} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{80}^{17} & 0 & 0 \end{pmatrix}\Theta(\tau).$$

Let $C = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, then $C \in O_A^+(\mathbf{Z})$, $Cc_1 = c_2$ and $Cc_2 = c_1$. Hence, using Corollary 2.15, we find

$$\vartheta_{a,b}^{c_1,c_2} = \vartheta_{Ca,Cb}^{c_2,c_1} = -\vartheta_{Ca,Cb}^{c_1,c_2}$$

Using this and Corollary 2.9, we obtain

$$\Theta\Big(-\frac{1}{\tau}\Big) = -i\tau\frac{2}{\sqrt{5}}\, M_5\; \Theta(\tau).$$

If we use $\eta(\tau+1) = \zeta_{24}\eta(\tau)$ and $\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\;\eta(\tau)$, we obtain the transformations for $H_{5,1}$.

If we write $\rho(\nu;\tau)$ as the sum of the three expressions (2.7), (2.8) and (2.9) and use Proposition 4.3, we find

$$\vartheta_{\left(\begin{smallmatrix}1/10\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)}(\tau)=2\eta(\tau)q^{-\frac{1}{60}}f_0(q)-2\eta(\tau)\left(\zeta_{12}R_{\frac{1}{30},\frac{5}{2}}(30\tau)+\zeta_{12}^{-1}R_{\frac{11}{30},\frac{5}{2}}(30\tau)\right).$$

Hence

$$\frac{1}{2\eta(\tau)}\,\vartheta_{\left(\begin{smallmatrix}1/10\\0\end{smallmatrix}\right),\left(\begin{smallmatrix}0\\1/4\end{smallmatrix}\right)}(\tau)=q^{-\frac{1}{60}}f_0(q)-\left(\zeta_{12}R_{\frac{1}{30},\frac{5}{2}}(30\tau)+\zeta_{12}^{-1}R_{\frac{11}{30},\frac{5}{2}}(30\tau)\right).$$

We can find similar identities for the other components of $H_{5,1}$. Combining them gives

$$H_{5,1}=F_{5,1}-G_{5,1}.$$

Using (5) of Proposition 4.2 and the fact that $F_{5,1}$ is holomorphic, we find

$$\Omega_{\frac{1}{2}}G_{5,1}=\frac{3}{16}G_{5,1}$$
$$\Omega_{\frac{1}{2}}F_{5,1}=\frac{3}{16}F_{5,1},$$

and so

$$\Omega_{\frac{1}{2}}H_{5,1}=\frac{3}{16}H_{5,1}.$$

Hence $H_{5,1}$ is a vector valued real-analytic modular form of weight 1/2. From (3) of Proposition 4.2 we get that $G_{5,1}$ is bounded as $\tau\downarrow\xi$, with $\xi\in\mathbf{Q}$. □

As with Corollary 4.6, we could also use Proposition 4.10 to describe the non-modularity of the "fifth order" mock ϑ-function $F_{5,1}$. We omit this.

We turn now to the other four "fifth order" mock ϑ-functions.

Lemma 4.11 *We have*

$$2\zeta_{12}^{-1}\frac{\eta(\tau)^2}{\eta(2\tau)}\,q^{-\frac{1}{60}}\psi_0(q)=\sum_{\nu\in\left(\begin{smallmatrix}1/10\\1/6\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/6\end{smallmatrix}\right)\right)}$$

$$2\zeta_{12}^{-1}\frac{\eta(\tau)^2}{\eta(2\tau)}\,q^{\frac{11}{60}}\psi_1(q)=\sum_{\nu\in\left(\begin{smallmatrix}3/10\\1/6\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\operatorname{sgn}(B(\nu,c_1))-\operatorname{sgn}(B(\nu,c_2))\Big\}\,e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/6\end{smallmatrix}\right)\right)}$$

$$2\zeta_{60}\frac{\eta(\tau)^2}{\eta(\tau/2)}\ q^{-\frac{1}{240}}\varphi_0(-q^{\frac{1}{2}}) = \sum_{\nu\in\left(\begin{smallmatrix}1/5\\1/6\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\mathrm{sgn}(B(\nu,c_1))-\mathrm{sgn}(B(\nu,c_2))\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}1/10\\1/6\end{smallmatrix}\right)\right)}$$

$$-2\zeta_{60}^{7}\frac{\eta(\tau)^2}{\eta(\tau/2)}\ q^{-\frac{49}{240}}\varphi_1(-q^{\frac{1}{2}}) = \sum_{\nu\in\left(\begin{smallmatrix}2/5\\1/6\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\mathrm{sgn}(B(\nu,c_1))-\mathrm{sgn}(B(\nu,c_2))\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}1/10\\1/6\end{smallmatrix}\right)\right)}$$

$$2\zeta_{16}^{-1}\frac{\eta(\tau)^2}{\eta((\tau+1)/2)}\ q^{-\frac{1}{240}}\varphi_0(q^{\frac{1}{2}}) = \sum_{\nu\in\left(\begin{smallmatrix}1/5\\1/6\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\mathrm{sgn}(B(\nu,c_1))-\mathrm{sgn}(B(\nu,c_2))\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/6\end{smallmatrix}\right)\right)}$$

$$2\zeta_{16}^{-1}\frac{\eta(\tau)^2}{\eta((\tau+1)/2)}\ q^{-\frac{49}{240}}\varphi_1(q^{\frac{1}{2}}) = \sum_{\nu\in\left(\begin{smallmatrix}2/5\\1/6\end{smallmatrix}\right)+\mathbf{Z}^2}\Big\{\mathrm{sgn}(B(\nu,c_1))-\mathrm{sgn}(B(\nu,c_2))\Big\}\ e^{2\pi iQ(\nu)\tau+2\pi iB\left(\nu,\left(\begin{smallmatrix}0\\1/6\end{smallmatrix}\right)\right)},$$

with $A=\left(\begin{smallmatrix}5&0\\0&-3\end{smallmatrix}\right)$, $c_1=\left(\begin{smallmatrix}3\\5\end{smallmatrix}\right)$ *and* $c_2=\left(\begin{smallmatrix}-3\\5\end{smallmatrix}\right)$.

Remark 4.12 We have $B(c_1,c_2)=-120$ and $Q(c_1)=Q(c_2)=-15$. If we choose C_Q such that $c_1\in C_Q$ then also $c_2\in C_Q$.

Proof: The proof is similar to the proof of Lemma 4.8. We also have to use

$$\frac{(q^{\frac{1}{2}})_\infty}{(q)_\infty}=\frac{(q^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}{(q)_\infty^2}=q^{\frac{1}{16}}\frac{\eta(\tau/2)}{\eta(\tau)^2}$$
$$\frac{(-q^{\frac{1}{2}})_\infty}{(q)_\infty}=\zeta_{48}^{-1}q^{\frac{1}{16}}\frac{\eta\big((\tau+1)/2\big)}{\eta(\tau)^2},$$

and

$$\frac{(-q)_\infty}{(q)_\infty}=\frac{(q^2;q^2)_\infty}{(q)_\infty^2}=\frac{\eta(2\tau)}{\eta(\tau)^2}.$$

Only the first identity is a bit more difficult: We have

$$\begin{aligned}
-\sum_{\substack{n=1\\|j|<n}}^{\infty}&(-1)^j q^{\frac52 n^2-\frac12 n-\frac32 j^2-\frac12 j}(1-q^n)\\
&=\sum_{\substack{n=1\\|j|<n}}^{\infty}(-1)^j q^{\frac52 n^2+\frac12 n-\frac32 j^2-\frac12 j}-\sum_{\substack{n=1\\|j|<n}}^{\infty}(-1)^j q^{\frac52 n^2-\frac12 n-\frac32 j^2-\frac12 j}\\
&=\Big(\sum_{n+j>0,n-j>0}-\sum_{n+j<0,n-j<0}\Big)(-1)^j q^{\frac52 n^2+\frac12 n-\frac32 j^2-\frac12 j},
\end{aligned}$$

where we have replaced n by $-n$ in the second sum. Hence

$$\begin{aligned}
1+2\sum_{n=1}^{\infty}&(-1)^n q^{n^2+n}-2\sum_{\substack{n=1\\|j|<n}}^{\infty}(-1)^j q^{\frac52 n^2-\frac12 n-\frac32 j^2-\frac12 j}(1-q^n)\\
&=2\Big(\sum_{n+j\geq 0,n-j>0}-\sum_{n+j<0,n-j\leq 0}\Big)(-1)^j q^{\frac52 n^2+\frac12 n-\frac32 j^2-\frac12 j}+2\sum_{n=-\infty}^{-1}(-1)^n q^{n^2}+1.
\end{aligned}\tag{4.6}$$

We have

$$2\sum_{n=-\infty}^{-1}(-1)^n q^{n^2}+1=\sum_{n\in\mathbf{Z}}(-1)^n q^{n^2}=\frac{(q)_\infty}{(-q)_\infty}.$$

Using this and (4.6) we find

$$\psi_0(q)=\frac{(-q)_\infty}{(q)_\infty}\Big(\sum_{n+j\geq 0,n-j>0}-\sum_{n+j<0,n-j\leq 0}\Big)(-1)^j q^{\frac52 n^2+\frac12 n-\frac32 j^2-\frac12 j},$$

from which the result follows. □

We collect these six mock ϑ-functions into a single vector-valued mock ϑ-function

$$F_{5,2}(\tau):=\begin{pmatrix}2q^{-\frac{1}{60}}\psi_0(q)\\ 2q^{\frac{11}{60}}\psi_1(q)\\ q^{-\frac{1}{240}}\varphi_0(-q^{\frac12})\\ -q^{-\frac{49}{240}}\varphi_1(-q^{\frac12})\\ q^{-\frac{1}{240}}\varphi_0(q^{\frac12})\\ q^{-\frac{49}{240}}\varphi_1(q^{\frac12})\end{pmatrix}$$

To express its modular transformation behaviour, we also introduce

$$H_{5,2}(\tau) := \frac{1}{2\eta(\tau)^2} \begin{pmatrix} 2\zeta_{12}\ \eta(2\tau)\ \vartheta_{\left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)}(\tau) \\ 2\zeta_{12}\ \eta(2\tau)\ \vartheta_{\left(\begin{smallmatrix} 3/10 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)}(\tau) \\ \zeta_{60}^{-1}\eta(\tau/2)\ \vartheta_{\left(\begin{smallmatrix} 1/5 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right)}(\tau) \\ \zeta_{60}^{-7}\eta(\tau/2)\ \vartheta_{\left(\begin{smallmatrix} 2/5 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right)}(\tau) \\ \zeta_{16}\ \eta\big((\tau+1)/2\big)\ \vartheta_{\left(\begin{smallmatrix} 1/5 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)}(\tau) \\ \zeta_{16}\ \eta\big((\tau+1)/2\big)\ \vartheta_{\left(\begin{smallmatrix} 2/5 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)}(\tau) \end{pmatrix},$$

with A, c_1 and c_2 as in Lemma 4.11, and

$$G_{5,2}(\tau) := -\frac{1}{2} \begin{pmatrix} 2\zeta_{12}R_{\frac{1}{30},\frac{5}{2}} + 2\zeta_{12}^{-1}R_{\frac{11}{30},\frac{5}{2}} \\ 2\zeta_{12}R_{\frac{13}{30},\frac{5}{2}} + 2\zeta_{12}^{-1}R_{\frac{23}{30},\frac{5}{2}} \\ -R_{\frac{19}{60},0} - R_{\frac{29}{60},0} + R_{\frac{49}{60},0} + R_{\frac{59}{60},0} \\ -R_{\frac{13}{60},0} - R_{\frac{23}{60},0} + R_{\frac{43}{60},0} + R_{\frac{53}{60},0} \\ \zeta_{24}^{-5}R_{\frac{19}{60},\frac{5}{2}} + \zeta_{24}^{5}R_{\frac{29}{60},\frac{5}{2}} + \zeta_{24}R_{\frac{49}{60},\frac{5}{2}} + \zeta_{24}^{-1}R_{\frac{59}{60},\frac{5}{2}} \\ \zeta_{24}R_{\frac{13}{60},\frac{5}{2}} + \zeta_{24}^{-1}R_{\frac{23}{60},\frac{5}{2}} + \zeta_{24}^{-5}R_{\frac{43}{60},\frac{5}{2}} + \zeta_{24}^{5}R_{\frac{53}{60},\frac{5}{2}} \end{pmatrix}(30\tau).$$

Proposition 4.13 *We have*

$$F_{5,2} = H_{5,2} + G_{5,2},$$

where

(1) *The function $H_{5,2}$ is a (vector-valued) real-analytic modular form of weight 1/2, satisfying*

$$H_{5,2}(\tau+1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{60}^{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{240}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{240}^{71} \\ 0 & 0 & \zeta_{240}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{240}^{71} & 0 & 0 \end{pmatrix} H_{5,2}(\tau),$$

and

$$H_{5,2}\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\frac{2}{\sqrt{5}}\ M_5\ H_{5,2}(\tau),$$

with M_5 as in Proposition 4.10, and is an eigenfunction of the Casimir operator $\Omega_{\frac{1}{2}}$, with eigenvalue $\frac{3}{16}$.

(2) *The function $G_{5,2}$ is bounded if $\tau \downarrow \xi$, with $\xi \in \mathbf{Q}$.*

Proof: We consider the function

$$\Theta(\tau) = \begin{pmatrix} \vartheta_{\left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix} 3/10 \\ 1/6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix} 1/5 \\ 1/6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix} 2/5 \\ 1/6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix} 1/5 \\ 1/6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)} \\ \vartheta_{\left(\begin{smallmatrix} 2/5 \\ 1/6 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)} \end{pmatrix} (\tau).$$

Using (4) and (2) of Corollary 2.9 we see

$$\Theta(\tau+1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{60}^{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{120}^{19} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{120}^{67} \\ 0 & 0 & \zeta_{24}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{120}^{19} & 0 & 0 \end{pmatrix} \Theta(\tau).$$

Let $C = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, then $C \in O_A^+(\mathbf{Z})$, $Cc_1 = c_2$ and $Cc_2 = c_1$. Hence, using Corollary 2.15, we find

$$\vartheta_{a,b}^{c_1,c_2} = \vartheta_{Ca,Cb}^{c_2,c_1} = -\vartheta_{Ca,Cb}^{c_1,c_2}$$

Using this and Corollary 2.9, we obtain

$$\Theta\Big(-\frac{1}{\tau}\Big) = -i\tau \frac{2}{\sqrt{5}} \cdot \begin{pmatrix} 0 & 0 & \zeta_{10}^{-1}\sin\frac{\pi}{5} & \zeta_5^{-1}\sin\frac{2\pi}{5} & 0 & 0 \\ 0 & 0 & \zeta_{10}^{-1}\sin\frac{2\pi}{5} & -\zeta_5^{-1}\sin\frac{\pi}{5} & 0 & 0 \\ \zeta_{10}\sin\frac{\pi}{5} & \zeta_{10}\sin\frac{2\pi}{5} & 0 & 0 & 0 & 0 \\ \zeta_5\sin\frac{2\pi}{5} & -\zeta_5\sin\frac{\pi}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\frac{2\pi}{5} & \sin\frac{\pi}{5} \\ 0 & 0 & 0 & 0 & \sin\frac{\pi}{5} & -\sin\frac{2\pi}{5} \end{pmatrix} \Theta(\tau).$$

If we use that

$$f(\tau) := \begin{pmatrix} \eta(\tau/2) \\ \eta\big((\tau+1)/2\big) \\ \eta(2\tau) \end{pmatrix}$$

transforms as

$$f(\tau+1) = \begin{pmatrix} 0 & 1 & 0 \\ \zeta_{24} & 0 & 0 \\ 0 & 0 & \zeta_{12} \end{pmatrix} f(\tau)$$

$$f\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau} \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} f(\tau),$$

we obtain the transformations for $H_{5,2}$.

If we write $\rho(\nu;\tau)$ as the sum of the three expressions (2.7), (2.8) and (2.9) and use Proposition 4.3, we find

$$\vartheta_{\left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)}(\tau) = 2\zeta_{12}^{-1}\frac{\eta(\tau)^2}{\eta(2\tau)}\, q^{-\frac{1}{60}}\psi_0(q) + \frac{\eta(\tau)^2}{\eta(2\tau)}\Big(R_{\frac{1}{30},\frac{5}{2}}(30\tau) + \zeta_6^{-1}R_{\frac{11}{30},\frac{5}{2}}(30\tau)\Big).$$

Hence

$$\zeta_{12}\frac{\eta(2\tau)}{\eta(\tau)^2}\,\vartheta_{\left(\begin{smallmatrix} 1/10 \\ 1/6 \end{smallmatrix}\right),\left(\begin{smallmatrix} 0 \\ 1/6 \end{smallmatrix}\right)}(\tau) = 2q^{-\frac{1}{60}}\psi_0(q) + \zeta_{12}R_{\frac{1}{30},\frac{5}{2}}(30\tau) + \zeta_{12}^{-1}R_{\frac{11}{30},\frac{5}{2}}(30\tau).$$

We can find similar identities for the other components of $H_{5,2}$. Combining them gives

$$H_{5,2} = F_{5,2} - G_{5,2}.$$

Using (5) of Proposition 4.2 and the fact that $F_{5,2}$ is holomorphic, we find

$$\Omega_{\frac{1}{2}}G_{5,2} = \frac{3}{16}G_{5,2}$$
$$\Omega_{\frac{1}{2}}F_{5,2} = \frac{3}{16}F_{5,2},$$

and so

$$\Omega_{\frac{1}{2}}H_{5,2} = \frac{3}{16}H_{5,2}.$$

Hence $H_{5,2}$ is a vector valued real-analytic modular form of weight 1/2. From (3) of Proposition 4.2 we get that $G_{5,2}$ is bounded as $\tau \downarrow \xi$, with $\xi \in \mathbf{Q}$. □

Proposition 4.14 *The (holomorphic) function $F_5 := F_{5,1} + F_{5,2}$ is a (vector-valued) modular form of weight 1/2, satisfying*

$$F_5(\tau+1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_{60}^{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_{240}^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & \zeta_{240}^{71} \\ 0 & 0 & \zeta_{240}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{240}^{71} & 0 & 0 \end{pmatrix} F_5(\tau),$$

and

$$F_5\Big(-\frac{1}{\tau}\Big) = \sqrt{-i\tau}\frac{2}{\sqrt{5}}\ M_5\ F_5(\tau),$$

with M_5 as in Proposition 4.10.

Proof: Since $G_{5,2} = -G_{5,1}$, we get from Proposition 4.10 and Proposition 4.13

$$F_5 = H_{5,1} + H_{5,2}.$$

Using this, the transformation behaviour of F follows directly from the transformation behaviour of $H_{5,1}$ and $H_{5,2}$ given in Proposition 4.10 and Proposition 4.13. □

Proposition 4.14 implies that each of the six components of $F_5 := F_{5,1} + F_{5,2}$, i.e. each of the six functions

$$\begin{gathered} q^{-\frac{1}{60}} f_0(q) + 2q^{-\frac{1}{60}}\psi_0(q) \\ q^{\frac{11}{60}} f_1(q) + 2q^{\frac{11}{60}}\psi_1(q) \\ q^{-\frac{1}{240}}(-1 + F_0(q^{\frac{1}{2}})) + q^{-\frac{1}{240}}\varphi_0(-q^{\frac{1}{2}}) \\ q^{\frac{71}{240}} F_1(q^{\frac{1}{2}}) - q^{-\frac{49}{240}}\varphi_1(-q^{\frac{1}{2}}) \\ q^{-\frac{1}{240}}(-1 + F_0(-q^{\frac{1}{2}})) + q^{-\frac{1}{240}}\varphi_0(q^{\frac{1}{2}}) \\ q^{\frac{71}{240}} F_1(-q^{\frac{1}{2}})) + q^{-\frac{49}{240}}\varphi_1(q^{\frac{1}{2}}) \end{gathered}$$

is a modular form on a suitable congruence subgroup of $\mathrm{SL}_2(\mathbf{Z})$. The first four functions were already given in terms of theta functions in [27, pp. 299]. Similar identities for the fifth and sixth function follow directly from the identities for the third and fourth function by replacing $q^{\frac{1}{2}}$ by $-q^{\frac{1}{2}}$.

4.5 Other mock ϑ-functions

In the previous sections we have dealt with the seventh order and most of the fifth order mock ϑ-functions from Ramanujan's letter. We were able to do this because identities for these mock ϑ-functions were available in the literature. Similar identities for the other two fifth order mock ϑ-functions, χ_0 and χ_1, are not available in the literature, as far as I know. I have found the following identities (which I will not prove here) for χ_0 and χ_1

$$\chi_0(q) = 2 - \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m} q^{\frac{1}{2}k^2+\frac{1}{2}l^2+\frac{1}{2}m^2+2kl+2km+2lm+\frac{1}{2}(k+l+m)},$$

$$\chi_1(q) = \frac{1}{(q)_\infty^2}\Big(\sum_{k,l,m\geq 0} + \sum_{k,l,m<0}\Big)(-1)^{k+l+m} q^{\frac{1}{2}k^2+\frac{1}{2}l^2+\frac{1}{2}m^2+2kl+2km+2lm+\frac{3}{2}(k+l+m)}.$$

These series are similar to the ones we used for the seventh and fifth order mock theta functions. However, the quadratic form is of type $(1, 2)$. Hence we cannot apply any of the results from Chapter 2. However, in [27] identities for χ_0 and χ_1 are given which give χ_0 and χ_1 as a linear combination of other fifth order mock ϑ-functions. Hence, we could derive the transformation properties of χ_0 and χ_1.

In Ramanujan's letter four third order mock ϑ-functions are given. Watson (see [26]) defined three more third order mock ϑ-functions. Later even more exotic mock ϑ-functions were introduced: of sixth order (see [5]), of eighth order (see [10]) and of tenth order (see [6], [7] and [8]). In these articles, identities are given, which relate the mock ϑ-functions to sums of the same type as the ones we used for the fifth and seventh order mock ϑ-functions. Hence, using the same techniques, we could derive the transformation properties of these mock ϑ-functions. In [30] I derive the transformation properties of the vector-valued third order mock ϑ-function

$$F(\tau) = \begin{pmatrix} q^{-\frac{1}{24}} f(q) \\ 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}) \\ 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}) \end{pmatrix}.$$

The result is similar to the results found in this chapter.