Introduction

Mock ϑ-functions

Early in 1920, three months before his death, S. Ramanujan wrote his last letter to G.H. Hardy. For the mathematical part of this letter see [21, pp. 127–131] (also reproduced in [4]). In the course of it he said: "I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper [22]) they enter into mathematics as beautifully as the ordinary ϑ-functions. I am sending you with this letter some examples." He then provided a long list of mock ϑ-functions, together with identities satisfied by them. The first three pages in which Ramanujan explained what he meant by a mock ϑ-function are very obscure.

G.N. Watson wrote the first papers ([26] and [27]) to elucidate the mock ϑ-functions. The first of these is Watson's Presidential Address to the London Mathematical Society in 1935. He entitled it "The Final Problem: An Account of the Mock Theta Functions." In it he writes: "I make no apologies for my subject being what is now regarded as old-fashioned, because, as a friend remarked to me a few months ago, I am an old-fashioned mathematician." His methods may have been a bit old-fashioned, but looking at the number of articles on mock ϑ-functions that have appeared since 1935, or even in the last ten years, we must conclude that the subject is still up-to-date.

In these two papers, Watson proves most of the assertions found in the letter of Ramanujan. The first paper considers only the third-order functions. It provides three new mock ϑ-functions not mentioned in the letter. The bulk of the paper is devoted to the modular transformation properties of these functions. To get these transformations, he first proves certain identities. For example, for the third order mock ϑ-function

$$f(q) := 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \frac{q^9}{(1+q)^2(1+q^2)^2(1+q^3)^2} + \cdots$$

he finds:

$$f(q) = \frac{2}{(q)_\infty} \sum_{n\in\mathbf{Z}} \frac{(-1)^n q^{\frac{3}{2}n^2+\frac{1}{2}n}}{1+q^n}, \tag{1}$$

with $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H} := \{\tau \in \mathbf{C} \mid \mathrm{Im}(\tau) > 0\}$, and $(q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = q^{-\frac{1}{24}}\eta(\tau)$, where η is the Dedekind eta-function.

In Watson's second paper on mock ϑ-functions, he moves on to the fifth order functions. He manages to prove all of the identities given by Ramanujan in his letter. However, he is unable to find the modular transformation properties, simply because he is unable to find identities like (1) for the fifth order functions. He even expressed his doubts about finding anything comparable to (1). However Andrews (see [2]) was able to find comparable results for most of the fifth order functions. For example, for the fifth order function which Watson denotes by f_0 one finds

$$f_0(q) = \frac{1}{(q)_\infty} \sum_{n \geq 0} \sum_{|j| \leq n} (-1)^j q^{\frac{5}{2}n^2 + \frac{1}{2}n - j^2}(1 - q^{4n+2}),$$

which may be rewritten as

$$f_0(q) = \frac{1}{(q)_\infty} \left(\sum_{n+j \geq 0, n-j \geq 0} - \sum_{n+j<0, n-j<0} \right) (-1)^j q^{\frac{5}{2}n^2 + \frac{1}{2}n - j^2}. \tag{2}$$

The seventh order functions were mostly neglected by Watson, perhaps because Ramanujan makes no positive assertions about them. However A. Selberg (see [23]) provides a full account of the behaviour of the seventh order functions near the unit circle. In [12, pp. 666] we find identities similar to (2) for the seventh order mock ϑ-functions $\mathcal{F}_0$, $\mathcal{F}_1$ and $\mathcal{F}_2$. For example (slightly rewritten)

$$\mathcal{F}_0(q) = \frac{1}{(q)_\infty} \left(\sum_{r,s \geq 0} - \sum_{r,s<0} \right) (-1)^{r+s} q^{\frac{3}{2}r^2 + 4rs + \frac{3}{2}s^2 + \frac{1}{2}r + \frac{1}{2}s}. \tag{3}$$

In [11] L. Göttsche and D. Zagier consider sums like the ones in (2) and (3). They call them theta-functions for indefinite lattices. For some special cases they find modular transformation properties for these functions. However, these results do not include the sums in (2) and (3). In [20], a theorem about the modularity of a certain family of q-series associated with indefinite binary quadratic forms is given. Again, the results do not include the sums in (2) and (3).

This thesis

This thesis is the result of my research on the following two questions, both posed by Don Zagier:

1. How do the mock ϑ-functions fit in the theory of modular forms?

2. Is there a theory of indefinite theta functions?

Since most of the mock ϑ-functions had been related to sums like the one in (1), I first considered this type of sum. The result of this research is Chapter 1. In it we consider the series

$$\sum_{n\in\mathbf{Z}} \frac{(-1)^n e^{\pi i(n^2+n)\tau+2\pi inv}}{1-e^{2\pi in\tau+2\pi iu}} \qquad (\tau\in\mathcal{H}, v\in\mathbf{C}, u\in\mathbf{C}\setminus(\mathbf{Z}\tau+\mathbf{Z})).$$

This function was also studied by Lerch in [15] (see [14] for an abstract). Therefore we call this a Lerch sum. This sum is of the same type as the sum in (1). The function does not transform like a Jacobi form. However, we find that on addition of a (relatively easy) correction term the function does transform like a Jacobi form. This correction term is real-analytic.

In Chapter 2 we consider certain indefinite ϑ-functions, in an attempt to give a partial answer to the second question. These indefinite ϑ-functions are modified versions of the sums considered by Göttsche and Zagier in [11]. We find elliptic and modular transformation properties for these functions. Because of the modifications the indefinite ϑ-functions are no longer holomorphic (in general). Although the results in this chapter are more general than the results in [11], the second question is far from being solved. This is because we only consider indefinite quadratic forms of type $(r-1,1)$. It remains a problem of considerable interest to develop a theory of theta-series for quadratic forms of arbitrary type.

In [3] Andrews gives most of the fifth order mock theta functions as Fourier coefficients of meromorphic Jacobi forms, namely certain quotients of ordinary Jacobi theta-series. This is the motivation for the study of the modularity of Fourier coefficients of meromorphic Jacobi forms, in Chapter 3. We find that modularity follows on adding a real-analytic correction term to the Fourier coefficients.

In Chapter 4 we use the results from Chapter 2, together with (2), (3) and similar identities for other mock ϑ-functions, to get the modular transformation properties of the seventh-order mock ϑ-functions and most of the fifth-order functions. The final result is that we can write each of these mock ϑ-functions as the sum of two functions H and G, where:

- H is a real-analytic modular form of weight $1/2$ and is an eigenfunction of the appropriate Casimir operator with eigenvalue $3/16$ (this is also the eigenvalue of holomorphic modular forms of this weight; for the theory of real-analytic modular forms see for example [16, Ch. IV]); and

- G is a theta series associated to a negative definite unary quadratic form, i.e. has the form $\sum \operatorname{sgn}(f)\beta(2f^2y)e^{-\pi if^2\tau-2\pi ifb}$, where f ranges over a certain arithmetic progression $a\mathbf{Z}+b$ $(a,b\in\mathbf{Q})$, $\tau=x+iy\in\mathcal{H}$ and $\beta(x)=\int_x^\infty u^{-\frac{1}{2}}e^{-\pi u}du$. Moreover G is bounded as τ tends vertically to any rational limit.

This decomposition is thus similar to the one found in [29] for an Eisenstein series of weight $3/2$, the holomorphic part of the series, in that case, having class numbers as Fourier coefficients.

Many of the results of Chapter 4 could also be deduced using the methods from Chapter 1 or Chapter 3 instead of Chapter 2, i.e. we have actually given 3 approaches to proving modularity properties of the mock ϑ-functions.