Chapter V

Optimal consumption in semimartingale markets

Consider an agent with finite horizon whose trading is constrained and who operates in a market where the prices of the assets are modelled as semimartingales. In this setting, we establish an existence result for maximising the expected utility of the agent over attainable inter-temporal consumption and final wealth. To obtain existence we do not resort to convex duality methods. We also give a characterisation result of the optimal solution.

1 Introduction

A general issue studied in economic sciences is the behaviour of agents in financial markets. In both classical and modern theory, one uses utility functions to model the different levels of "satisfaction" of the agents corresponding to different distributions of wealth over intermediate consumption and trading in risky assets. In two seminal papers [96, 98], Merton was the first to study this problem in a continuous time framework. He assumed that the market was driven by Markov state price processes. In this setting he could use dynamic programming and the Bellman equation to derive a partial differential equation for the value function, which he solved explicitly in the case of constant coefficients.

More recently, Cox and Huang, Pliska and Karatzas and co-authors developed in several papers, e.g. [43, 73, 109], a martingale approach to the problem of utility maximisation. In a *complete* market setting— that is, all the admissible stochastic claims can be replicated by the traded assets— the problem is decomposed into two steps. Firstly, a variational problem is solved. See for a close study of this type of optimisation problems [19]. Secondly, a portfolio financing this consumption-final wealth plan is found by using a martingale representation theorem. The *incomplete* market setting is considerably more complicated. In Karatzas et al. [74] and He and Pearson [69] the incomplete market was studied in a continuous time diffusion setting. In these papers the central idea is to solve a dual variational problem and to find the solution of the original problem by convex duality methods. Then, in the paper [84], Kramkov and Schachermayer also employed a duality approach to solve the problem of maximisation of the utility of final wealth in a general semimartingale market. Recently, Karatzas and Žitković [76] have taken up the line of [84] and extended it to the setting where also cone restrictions are put on the agent's trading strategies, where the agent has random endowment and where the agent not only gets utility from final wealth but also from intermediate consumption.

In this paper, we study the problem of optimal consumption in the same semimartingale setting with constrained trade. However, following the line set out by Cuoco [45] in the continuous time diffusion setting, we use a direct primal approach to obtain existence, without resorting to duality methods. Although we could have followed Cuoco in using Levin [90]'s technique of *relaxation-projection* to obtain existence, our existence proof relies on a famous result of Komlós [80]. Compared to Cuoco [45] and Karatzas and Žitković [76], our utility function does not need to be differentiable or increasing in the consumption.

The rest of the paper is organised as follows. In Section 2 we formulate the model. In Section 3 we show the control problem can be equivalently reformulated as a static variational problem and in 4 existence is obtained. Section 5 studies the characterisation of optimal policies and in Section 6, we consider two specific examples: in Section 6.1 we consider the case of a jump-diffusion market and in Section 6.2, we specialise to the the case of p-integrable consumption-final wealth plans in a complete markets with no restrictions on the trading.

2 Formulation of the model

2.1 Setting

In this paper we consider the continuous time model with a finite time horizon $T > 0$. Let $S = (S_1, \dots, S_n)$ be an n-dimensional semimartingale on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = \{\mathcal{F}_t : 0 \le t \le T\}, P)$ which satisfies the usual conditions (i.e. $\mathbf{F}$ is complete and right-continuous). The filtration $\mathbf{F}$ can be thought of as a model for the information structure of the market. Except for the integrands all processes which occur are assumed to be adapted to the filtration $\mathbf{F}$ and to have càdlàg paths (right-continuous paths with left limits).

Consider now a securities market where a bond and n risky assets are traded. The price process of the ith risky asset is given by $S_i = \{S_i(t), 0 \le t \le T\}$. We model the price of the bond $S_0 = \{S_0(t), 0 \le t \le T\}$ by $S_0(t) = \exp(\int_0^t r(s)\mathrm{d}s)$, where $r = \{r(t), 0 \le t \le T\}$ is some non-negative $\mathbf{F}$-adapted process. The process r represents the (risk-less) interest rate. We assume that for some constant $R > 0$

$$\int_0^T r(s)\mathrm{d}s < R \qquad P\text{-a.e.} \tag{1}$$

In the sequel we write $\gamma^0 = S_0^{-1}$.

Denote by $\mathcal{P}^0$ the set of all probability measures that are equivalent to P on $\mathcal{F}_T$ and under which the discounted price processes $\gamma^0 S_i$ are local martingales for $i = 1, \dots, n$. We make the following assumption on our process $(S_0, S) = (S_0, S_1, \dots, S_n)$ to ensure the absence of arbitrage (see Proposition 1 below).

Assumption 1 $\mathcal{P}^0 \neq \emptyset$.

For any interval $I \subset [0,T]$, we denote by $\mathcal{L}^0_+(I \times \Omega)$ the set of all processes $b = \{b(t), t \in I\}$ that are $\mathbf{F}$-adapted and nonnegative. In above setting of the security market operates an economic agent who seeks to maximise utility through consumption and investment. The agent is endowed with an initial capital $w_0 > 0$ and receives a stochastic income flow $y \in \mathcal{L}^0_+([0,T] \times \Omega)$, for example as labour income. We assume that the process y satisfies for some constant $K > 0$

$$\int_0^T \gamma^0(t) y(t) \mathrm{d}t \leq K. \qquad P\text{-a.e.} \tag{2}$$

At time t the agent can choose to buy and consume an amount $c_i(t)$ of the commodity i $(i = 1, \dots, m)$ or to invest his/her money in the security market to generate a final wealth at time T or to be able to consume at some later time. A final wealth plan and w consumption plan c are elements of $\mathcal{L}^0_+(\{T\} \times \Omega)$ and $(\mathcal{L}^0_+([0,T] \times \Omega))^m$ respectively.

The agent's preferences for consumption-final wealth plans (c, w) are represented by a functional U:

$$U(c, w) = E\left[\int_0^T u(t, c(t)) \mathrm{d}t + v(w)\right], \tag{3}$$

where preferences of intermediate consumption c are expressed through a time-additive function $u : [0,T] \times \mathbf{R}^m_+ \times \Omega \to [-\infty, \infty)$ and those of final wealth w through a function $v : \mathbf{R}_+ \times \Omega \to [-\infty, \infty)$. For each $c \in \mathbf{R}^m_+$, the function $(t, \omega) \mapsto u(t, c, \omega)$ is $\mathbf{F}$-adapted; for each $\omega \in \Omega$, the function $(t, c) \mapsto u(t, c, \omega)$ is measurable with respect to $\mathcal{B}([0,T] \times \mathbf{R}^m_+)$, the Borel σ-algebra of $[0,T] \times \mathbf{R}^m_+$. Similarly, the function v is measurable with respect to $\mathcal{B}(\mathbf{R}_+) \times \mathcal{F}_T$. The functions u, v may be random, reflecting the agent's changes in preferences. The agent who is just interested in inter-temporal consumption or just in final wealth is included in the model by setting v respectively u equal to zero. Later on we will impose conditions on the utility functions u and v to ensure that the functional U is well defined.

2.2 Constraints on trading

We begin with recalling some notations and facts from the theory of stochastic integration. For a comprehensive treatment of the subject of stochastic integration we refer to Dellacherie and Meyer [48], Protter [111] and Jacod and Shiryaev [71]. Let $X = (X_1, \dots, X_n)$ be an n-dimensional semimartingale and let $\mathcal{L}(X)$ denote the space of all n-dimensional predictable processes integrable with respect to X. This space contains among others the locally bounded predictable processes. The stochastic integral of $H \in \mathcal{L}(X)$ with respect to X will be denoted as $\int H \mathrm{d}X = \int H_1 \mathrm{d}X_1 + \ldots + \int H_n \mathrm{d}X_n$, where $\int H_i \mathrm{d}X_i$ denotes the stochastic integral of H_i with respect to the real valued semimartingale X_i.

For $H \in \mathcal{L}(X)$ the stochastic integral $\int H \mathrm{d}X$ is again a semimartingale. Identifying two processes H, $\widetilde{H}$ if the integrals $\int H \mathrm{d}X$ and $\int \widetilde{H} \mathrm{d}X$ are indistinguishable, that is

$$P\left(\left\{\omega : \int_0^t H(s,\omega)\mathrm{d}X(s,w) = \int_0^t \widetilde{H}(s,\omega)\mathrm{d}X(s,\omega) \text{ for all } t \in [0,T]\right\}\right) = 1,$$

we write $\mathbf{L}(X)$ for the resulting space of equivalence classes. In the sequel we will slightly abuse notation by writing H if we mean the equivalence class of H. An integrand $H \in \mathbf{L}(X)$ is called *(locally) admissible* if $\int H \mathrm{d}X$ is (locally) bounded from below. The class of all admissible (resp. locally admissible) processes is denoted by $\mathbf{L}^a(X)$ (resp. $\mathbf{L}^a_{loc}(X)$). The *Emery distance* D between two real valued semimartingales Y and Z is defined as

$$D(Y,Z) = \sup_{U \in \mathcal{U}} E\left[\min\left\{1, \sup_{t \le T}\left|\int U \mathrm{d}Y - \int U \mathrm{d}Z\right|\right\}\right]$$

with the supremum taken over the set $\mathcal{U}$ of predictable real valued processes bounded in absolute value by one. For this metric the space of real valued semimartingales (up to distinguishability) is complete, see Emery [55]. Mémin [95] has shown that the space $\mathbf{L}(X)$ is complete with respect to the metric

$$d_X(H,G) = D\left(\int H \mathrm{d}X, \int G \mathrm{d}X\right). \tag{4}$$

From now on X denotes the n-dimensional semimartingale $X = \gamma^0 S$. We assume that trading takes place continuously in time and that there are no transaction costs. Let $H_0(t)$ and $H(t) = (H_i(t))_{i=1}^n$ denote the number of bonds and shares of different types (1 till n) the agent has in his/her portfolio at time t. Assume that $H_0 = \{H_0(t), 0 \le t \le T\}$ is progressively measurable with respect to the filtration $\mathbf{F}$. The trading of the agent is subject to certain restrictions (e.g. prohibition of short selling). To model these constraints, we follow [59] and impose the condition that H lies in a certain subset of $\mathbf{L}^a_{loc}(X)$. Let $\mathcal{H} \subseteq \mathbf{L}^a_{loc}(X)$ be a family of locally admissible integrands for X. We assume that $\mathcal{H}$ contains $H \equiv 0$, is closed in $\mathbf{L}^a_{loc}(X)$ with respect to the metric d_X given in (4) and is convex in the following sense: for any H and G in $\mathcal{H}$ and any predictable process $0 \le h \le 1$ the process $hG + (1-h)H$ belongs to $\mathcal{H}$. A strategy is called $\mathcal{H}$-constrained if $H \in \mathcal{H}$.

Example 1 *Using $\mathcal{H}$ various constraints on the choice of trading strategies can be modelled. For instance, in the following cases the set $\mathcal{H}$ satisfies our assumptions:*

(i) No constraints: $\mathcal{H} = \mathbf{L}^a_{loc}(X)$;

(ii) Short-selling of assets 1 to k is not allowed:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : H_i \geq 0, 1 \leq i \leq k\};$$

(iii) Assets 1 *to k are non-tradeable:*

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : H_i = 0, 1 \leq i \leq k\};$$

(iv) The strategy H has to lie in a closed convex set $A \subset \mathbf{R}^n$ *containing* 0*:*

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : H \in A\};$$

(v) Minimum capital requirements:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : \sum_{i=1}^{n} H_i X_i \geq L\},$$

for some $L \leq 0$.

(vi) Constraints on the amounts of money $H_i X_i$ *invested:*

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : (H_1 X_1, \ldots, H_n X_n) \in A\},$$

where $A \subset \mathbf{R}^n$ *is a convex closed set containing* 0.

(vii) Upper and lower bounds on the number of shares:

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : L_i \leq H_i \leq U_i, 1 \leq i \leq n\},$$

where $L_i \leq 0 \leq U_i$ *and* L, U *belong to* $\mathbf{L}^a_{loc}(X)$.

(viii) Upper and lower bounds on the amounts of money $H_i X_i$ *invested:*

$$\mathcal{H} = \{H \in \mathbf{L}^a_{loc}(X) : L_i \leq H_i X_i \leq U_i, 1 \leq i \leq n\},$$

where L, U *are as in (vii).*

Set $\mathcal{Y}$ equal to the family of semimartingales $\mathcal{Y} = \{\int H \mathrm{d}X : H \in \mathcal{H}\}$. For any $Y \in \mathcal{Y}$ and any probability measure $P^* \sim P$ we denote by $A^Y_{P^*}$ the compensator of Y under P^*. We restrict ourselves to the family $\mathcal{P}$ of measures $P^* \sim P$ under which any $Y \in \mathcal{Y}$ is a special semimartingale* and under which there exists an increasing predictable process $A^{\mathcal{Y}}_{P^*}$ with

$$A^{\mathcal{Y}}_{P^*}(t) := \operatorname*{ess\,sup}_{Y \in \mathcal{Y}} A^Y_{P^*}(t) < \infty \tag{5}$$

a.e. for all $t \in [0, T]$. Then, for any $P^* \in \mathcal{P}$, the process $A^{\mathcal{Y}}_{P^*}$ is the minimal upper bound for all predictable processes arising in the Doob-Meyer decomposition of the special semimartingales $Y \in \mathcal{Y}$ (called the *upper variation process* of Y, see [59]). Note that the set $\mathcal{P}^0$ of equivalent local martingale measures is contained in $\mathcal{P}$. Indeed, since under $P^0 \in \mathcal{P}^0$ any element in $\mathcal{Y}$ is a local martingale (cf. [2]), we have that $A^{\mathcal{Y}}_{P^0}$ is identically zero.

*A semimartingale X is a special semimartingale if it admits the decomposition $X_t = X_0 + M_t + A_t$, where M is a local martingale with $M_0 = 0$ and A is a predictable process which is locally of integrable variation with $A_0 = 0$

Example 2 *Let $P^* \sim P$. If the set $\mathcal{H}$ is a cone, the corresponding upper variation process $A^{\mathcal{Y}}_{P^*}$ is identically zero (since $A^{\lambda Y}_{P^*} = \lambda A^{Y}_{P^*}$ for $\lambda \geq 0$) and thus $\mathcal{P}$ is the set of measures $P^* \sim P$ under which any $Y \in \mathcal{Y}$ is a local supermartingale. Similarly, if $\mathcal{H}$ is a linear family, we find that the corresponding process $A^{\mathcal{Y}}_{P^*}$ is identically zero and $\mathcal{P}$ is the set of measures $P^* \sim P$ under which all $Y \in \mathcal{Y}$ are local martingales. As third example we consider $\mathcal{H}$ as in Example 1(vii). Let $P^* \sim P$ be a measure under which X is a special semimartingale and write $X = M + A$ for the canonical decomposition of X into a local martingale M under P^* and a predictable process A of bounded variation. For this class $\mathcal{H}$, one can then show, following [59], that*

$$A^{\mathcal{Y}}_{P^*}(t) = \sum_{i=1}^{n} \int_0^t U_i(s) \mathrm{d}A_i^+(s) - \sum_{i=1}^{n} \int_0^t L_i(s) \mathrm{d}A_i^-(s), \qquad t \geq 0,$$

(where $A = A^+ - A^-$ with $A^{\pm}$ predictable increasing processes) and that the set $\mathcal{P}$ contains the set of all $P^* \sim P$ under which X is a special semimartingale.

In the sequel we put the following restriction on the constraint family $\mathcal{H}$:

Assumption 2 *The family $\mathcal{H}$ is such that* $\sup_{P^* \in \mathcal{P}} E^*[A^{\mathcal{Y}}_{P^*}(T)] < \infty$.

We make this assumption for the ease and clarity of the presentation and without loss of generality. Indeed, since $A^{\mathcal{Y}}_{P^*}$ is locally bounded, Assumption 2 can be dispensed with using *localisation* arguments (see [111] for explanation and compare the proof of Proposition 4.2 in [59]). We omit the details.

2.3 The consumption problem

If the agent has followed trading strategy $(H_0, H) = (H_0, H_1, \ldots, H_n)$ up to time t, his/her wealth at time t is given by $\Pi(t) = H_0(t)S_0(t) + H(t) \cdot S(t)$ (where $\cdot$ denotes the inner product). For an agent with income y and initial wealth w_0, a pair (c, w) of a consumption plan $c \in \mathcal{L}^0_+([0,T] \times \Omega)$ and a final wealth $w \in \mathcal{L}^0_+(\{T\} \times \Omega)$ is called *$\mathcal{H}$-feasible* if there exists an $H \in \mathcal{H}$ and a non decreasing process $D \in \mathcal{L}^0_+([0,T] \times \Omega)$ such that Π is bounded from below, reaches the wealth w at time T, $\Pi(T) \geq w$ a.e., and satisfies for all $t \in [0,T]$

$$\Pi(t) = w_0 + \int_0^t H_0(s) \mathrm{d}S_0(s) + \int_0^t H(s) \mathrm{d}S(s) - C(t) \tag{6}$$

where

$$C(t) = \int_0^t \left(\sum_{i=1}^{m} c_i(s) - y(s) \right) \mathrm{d}s + D(t).$$

The set of all $\mathcal{H}$-feasible consumption-final wealth plans (c, w) is denoted by $\mathcal{A}^0(\mathcal{H})$. The process D in (6) covers the possibility of free disposal of wealth, that is the agent is allowed to reinvest not his entire wealth. The amount of wealth wasted up to time t is then given by $D(t)$. Equation (6) is the usual dynamic budget constraint. It states at time t the wealth is equal to the initial wealth plus trading gains minus net withdrawals. The problem facing the agent can now be stated as the following control problem:

$$\sup_{(c,w)\in\mathcal{A}^0(\mathcal{H})} U(c,w) = \sup_{(c,w)\in\mathcal{A}^0(\mathcal{H})} E\left[\int_0^T u(t,c(t))\mathrm{d}t + v(w)\right]. \qquad (\mathcal{V})$$

For the agent's optimisation problem $(\mathcal{V})$ to be well posed it is necessary that there are no *arbitrage possibilities* in the market attainable for the agent, that is, the agent can not make a risk-less profit using some $\mathcal{H}$-feasible trading strategy. To be more precise, an arbitrage possibility is a nonzero consumption-final wealth plan (c,w) that is $\mathcal{H}$-feasible for zero initial wealth w_0 and zero income y. To rule out strategies such as the doubling strategy of Harrison and Kreps [66], we imposed the condition that the wealth process Π is bounded from below.

Proposition 1 *Under Assumption 1 there are no arbitrage possibilities.*

Proof Use partial integration and recall that $\Pi = H_0S_0 + H\cdot S$ to find that (6) implies

$$\begin{aligned}\gamma^0(t)\Pi(t) + \int_0^t \gamma^0(s)\mathrm{d}C(s) &= -\int_0^t [H_0(s)S_0(s) + H(s)\cdot S(s)]r(s)\gamma^0(s)\mathrm{d}s \\ &\quad + \int_0^t \gamma^0(s)H(s)\mathrm{d}S(s) \\ &= \int_0^t H(s)\mathrm{d}(\gamma^0(s)S(s)) =: J_t \qquad (7)\end{aligned}$$

Since H is locally admissible, $J = \{J_t, t\in[0,T]\}$ is a local martingale under all measures $P^0 \in \mathcal{P}^0$. Since the left-hand side is bounded below, Fatou's lemma then implies that J is a supermartingale and we find that

$$E^0\left[\gamma^0(T)\Pi(T) + \int_0^T \gamma^0(s)\left(\sum_{i=1}^m c_i(s)\right)\mathrm{d}s\right] \le 0 \qquad \forall P^0\in\mathcal{P}^0.$$

Since $\gamma^0(t) \ge \exp(-R)$ for all $t\in[0,T]$, it follows that $(c,w) = (0,0)$ a.e. □

3 Reformulation

As in Cox and Huang [43] and Cuoco [45], the first step in establishing existence for $(\mathcal{V})$ is to reformulate the control problem $(\mathcal{V})$ as a static variational problem. We will write E^* for the expectation under the measure P^*. Recall the definition of the process $A^{\mathcal{Y}}_{P^*}$ given in (5).

Theorem 1 *Let $(c,w)\in(L^0_+([0,T]\times\Omega))^m \times \mathcal{L}^0_+(\{T\}\times\Omega)$ be a consumption-final wealth plan. Then the following two assertions are equivalent:*

(i) (c,w) *is* $\mathcal{H}$*-feasible;*

(ii) (c,w) *satisfies for all* $P^* \in \mathcal{P}$*:*

$$E^*\left[\int_0^T \gamma^0(t)\Big(\sum_{i=1}^m c_i(t) - y(t)\Big)\mathrm{d}t + \gamma^0(T)w\right] \leq w_0 + E^*\left[A^{\mathcal{Y}}_{P^*}(T)\right]. \quad (8)$$

Letting (c,w) a consumption-final wealth plan in $(L^0_+([0,T]\times\Omega))^m \times \mathcal{L}^0_+(\{T\}\times\Omega)$, we get from this result that there exists an $\mathcal{H}$-constrained strategy that the agent can follow to attain (c,w) if, and only if, (c,w) satisfies (8) for all $P^* \in \mathcal{P}$. Thus (c,w) is optimal for the control problem $(\mathcal{V})$ if, and only if, (c,w) is optimal for the variational problem

$$\sup_{(c,w)\in F} U(c,w), \quad (\mathcal{V}')$$

where F is the set of $(c,w) \in (L^0_+([0,T]\times\Omega))^m \times \mathcal{L}^0_+(\{T\}\times\Omega)$ that satisfy the inequality (8) for all $P^* \in \mathcal{P}$.

Proof $(i) \Rightarrow (ii)$ Let $\widetilde{H} \in \mathcal{H}$ be a strategy that implements (c,w), let P^* be any element of $\mathcal{P}$ and denote by $\widetilde{Y}$ the process $\widetilde{Y} = \int \widetilde{H}\mathrm{d}(\gamma^0 S)$. Since $A^{\widetilde{Y}}_{P^*}$ denotes the compensator of $\widetilde{Y}$, the process $\widetilde{Y} - A^{\widetilde{Y}}_{P^*}$ is a local martingale under P^*. Let $\mathcal{T}$ denote the set of $\mathbf{F}$-stopping times and write $(\tau_m)_m$ with $\tau_m \in \mathcal{T}$ for a fundamental sequence of the stopping times belonging to this local martingale. That is, the τ_m form an increasing sequence $\tau_m \uparrow T$ a.e. such that the stopped processes $\{(\widetilde{Y} - A^{\widetilde{Y}}_{P^*})(t\wedge\tau_m), t \geq 0\}$ are uniformly integrable martingales. Adapting the partial integration argument (7) for general initial wealth w_0 and income y and taking expectations we find that

$$E^*\left[\gamma^0(T\wedge\tau_m)\Pi(T\wedge\tau_m) + \int_0^{T\wedge\tau_m}\gamma^0(s)\mathrm{d}C(s)\right] = w_0 + E^*[A^{\widetilde{Y}}_{P^*}(T\wedge\tau_m)],$$

where we used that $E^*[(\widetilde{Y} - A^{\widetilde{Y}}_{P^*})(T\wedge\tau_m)] = \widetilde{Y}(0) - A^{\widetilde{Y}}_{P^*}(0) = 0$. Since the integrand on the left-hand side of the previous display is uniformly bounded from below, Fatou's lemma yields

$$E^*\left[\gamma^0(T)\Pi(T) + \int_0^T \gamma^0(s)\mathrm{d}C(s)\right] \leq w_0 + \liminf_{m\to\infty} E^*[A^{\widetilde{Y}}_{P^*}(T\wedge\tau_m)]. \quad (9)$$

By monotonicity and the definition of $A^{\mathcal{Y}}_{P^*}$ it follows that $A^{\widetilde{Y}}_{P^*}(t) \leq A^{\mathcal{Y}}_{P^*}(T)$ which combined with (9) yields (8). $(ii) \Rightarrow (i)$ Define the process $W = \{W(t), 0 \leq t \leq T\}$ by

$$W(t) = \operatorname*{ess\,sup}_{P^*\in\mathcal{P}}\left\{E^*\left[L(T) + \gamma^0(T)w - A^{\mathcal{Y}}_{P^*}(T)|\mathcal{F}_t\right] + A^{\mathcal{Y}}_{P^*}(t)\right\},$$

where $L = \{L(t), t \in [0,T]\}$ is the process given by $L(t) = \int_0^t \gamma^0(u)[\sum_{i=1}^m c_i(u) - y(u)]\mathrm{d}u$. From (8) and Assumption 2 combined with (2) we see that $E^*[L(T) +$

$\gamma^0(T)w - A^{\mathcal{Y}}_{P^*}(T)]$ is uniformly bounded above and below for all $P^* \in \mathcal{P}$, respectively. By Theorem 2.1.1 in [53], the process W is a supermartingale under each $P^* \in \mathcal{P}$ with a càdlàg modification. Assume W denotes this modification. Since $A^{\mathcal{Y}}_{P^*}$ is increasing, then also the process $W - A^{\mathcal{Y}}_{P^*}$ is a supermartingale under any $P^* \in \mathcal{P}$. Recall that $A^{\mathcal{Y}}_{P^*} \equiv 0$ if P^* is an equivalent local martingale measure for $\gamma^0 S$. Combining with Assumption 1, (1) and (2), we deduce that W is uniformly bounded from below. The Constrained Optional Decomposition Theorem of Föllmer and Kramkov ([59, Theorem 4.1]) then implies that there exists a process $H \in \mathcal{H}$ and an increasing process G such that $W = W(0) + \int H \mathrm{d}(\gamma^0 S) - G$. Consider now the value process $\Pi = S_0(W - L)$. Note that Π is uniformly bounded from below, has càdlàg paths and has final value $\Pi(T) = w$. Moreover, we find by partial integration that

$$\mathrm{d}\Pi = H\mathrm{d}S + \gamma^0(\Pi - H \cdot S)\mathrm{d}S_0 - S_0\mathrm{d}G - \mathrm{d}\widetilde{L}$$

where $\widetilde{L} = \int (c(s) - y(s))\mathrm{d}s$. Thus, Π satisfies (6) for the strategy $(H_0, H) = (\gamma^0(\Pi - H \cdot S), H)$ and $D \equiv \int S_0 \mathrm{d}G$. Since $H \in \mathcal{H}$ and H_0 is adapted with càdlàg paths (and hence is progressively measurable), we conclude that (c, w) is $\mathcal{H}$-feasible. □

4 Existence

To prove existence for $(\mathcal{V})$, we can restrict ourselves, by the equivalence result from the previous section, to the study of the variational problem $(\mathcal{V}')$. In order for the functional U in (3) to be well defined and to guarantee existence for $(\mathcal{V}')$, u and v have to satisfy a certain asymptotic growth condition. A similar condition, which goes back to [14], appeared in [19] in a different context. Fix some $P^{0\star} \in \mathcal{P}^0$ (which is non-empty by Assumption 1) and denote by $\xi^{0\star}$ the Radon-Nikodym derivative of $P^{0\star}$ with respect to P on $\mathcal{F}_T$ and write $\xi^{0\star}(t) = E[\xi^{0\star}|\mathcal{F}_t]$. Let λ be the Lebesgue measure on $[0, T]$.

$$\left\{ \begin{array}{c} \text{For every } \epsilon > 0 \text{ there exist } \psi^\epsilon \in \mathcal{L}^1(P^{0\star}), \phi^\epsilon \in \mathcal{L}^1(\lambda \times P^{0\star}) \\ \text{such that for } P\text{-a.e. } \omega \in \Omega \text{ and } \lambda\text{-a.e. } t \in [0,T] \\ v(w,\omega) \le \epsilon \xi^{0\star}(T,\omega)|w| + \xi^{0\star}(T,\omega)\psi^\epsilon(\omega) \text{ for all } w \in \mathbf{R}_+ \\ u(t,c,\omega) \le \epsilon \xi^{0\star}(t,\omega)|c| + \xi^{0\star}(t,\omega)\phi^\epsilon(t,\omega) \text{ for all } c \in \mathbf{R}^m_+ \end{array} \right\}. \qquad (\gamma_1)$$

Now we can state the main result of this paper:

Theorem 2 *Let u, v satisfy the asymptotic growth property (γ_1) and suppose $u(t, \cdot, \omega), v(\cdot, \omega)$ are concave. Then the problems $(\mathcal{V})$ and $(\mathcal{V}')$ have an optimal solution.*

Note that the problem $(\mathcal{V})$ is feasible. Indeed, first note that since $H \equiv 0 \in \mathcal{H}$ (it is allowed for the agent not to trade), by (5) the process $A^{\mathcal{Y}}_{P^*}$ is non-negative for any $P^* \in \mathcal{P}$. Assumption 1 and Theorem 1 imply then that

$(c, w) = (c_1, c_2, \ldots, w) = (y, 0, \ldots, 0)$ is $\mathcal{H}$-feasible for $(\mathcal{V})$. Moreover, if in addition u or v is strictly concave, the problem $(\mathcal{V})$ has a unique solution (where we identify two solutions that differ only on a null-set).†

As the following example shows, a considerable class of utility functions satisfy the conditions of Theorem 2.

Example 3 *Let $b \in (0,1)$, $k_1 \geq 0$ and $k_2 > 0$ and suppose $1/\xi^{0\star} \in \mathcal{L}^{1/b}(\lambda \times P^{0\star})$. Consider utility functions for intermediate consumption u which are non-decreasing and concave in c and satisfy for $\lambda \times P$-a.e. $(t,\omega) \in [0,T] \times \Omega$ the growth condition*

$$u(t, c, \omega) \leq k_1 + k_2 |c|^{1-b} \quad c \in \mathbf{R}_+^m. \tag{10}$$

Note this condition also appears in Cox and Huang [43] and Cuoco [45] for one-dimensional consumption. Then $u(z)$ is $o(|z|)$ as $|z| \to \infty$; that is, the function u is asymptotically dominated by $|z|$. The functions u defined by (10) satisfy growth property (γ_1), as follows from Example 4.2 and Proposition 4.3 in [19]. As concrete examples of utility functions satisfying the bound (10) we find the HARA *utility functions for m-dimensional consumption*

$$u(t, c) = e^{-\rho t} \frac{b}{1-b} \left(\frac{\alpha \cdot c}{b} + \beta \right)^{1-b} \qquad c \in \mathbf{R}_+^m$$

with $b \in (0, \infty) \backslash \{1\}, \alpha \in \mathbf{R}_+^m, \beta, \rho \geq 0$. If $b = 1$, $u(c,t) = e^{-\rho t} \log(\alpha \cdot c + \beta)$ and if $\beta = 1, b = \infty$, $u(c,t) = \exp(-\rho t - \alpha \cdot c)$.

Similarly, ignoring the time dependence and setting $m = 1$, one constructs examples of utility functions for final wealth. ◇

Remark. In our set-up also some optimisation problems connected to hedging can be incorporated. Consider an agent who has sold a contingent claim with pay-off V, a $\mathcal{F}_T$-measurable nonnegative random variable, at time T and now seeks to hedge against this claim. The trading of the agent is subject to certain restrictions which are incorporated in the set $\mathcal{H}$. Suppose that for this agent a super-hedge, a final wealth w that dominates V, is too expensive: there is no strategy $H \in \mathcal{H}$ such that the corresponding portfolio Π is bounded below and satisfies $\Pi(T) \geq V$ almost everywhere. The agent's attitude towards the risk of a shortfall $(V - w)^+$ is represented by the function

$$v(w, \omega) = -p^{-1}((V(\omega) - w)^+)^p, \qquad p \geq 1, w \geq 0.$$

The optimal final wealth for this agent is then given by the optimal solution of the corresponding control problem $(\mathcal{V})$. Since v only takes non-positive values and v is concave, Theorem 2 yields that there exists an optimal w_* for $(\mathcal{V})$ and an $\mathcal{H}$-constrained strategy which satisfies $\Pi(T) \geq w_*$ almost everywhere. See [60] for more on minimising shortfall in hedging problems.

†Similarly, in the setting of [76], only strict concavity in $t = T$ of the utility function is needed to get uniqueness

4.1 Proof of theorem 2

Firstly, we define the measure $\hat{\lambda} = \lambda + \delta_T$ on $[0, T]$ as a unit atom δ_T in $t = T$ added to the Lebesgue measure λ on $[0, T]$. Furthermore, we construct $\tilde{u}, \tilde{c}, \tilde{y}$ from u, v, c, y by setting $\tilde{u}, \tilde{c}, \tilde{y}$ on $[0, T)$ equal to u, c, y respectively and choosing $\tilde{u}(T, c)$, $\tilde{c}(T)$ and $\tilde{y}(T)$ to take the values $v(c_1)$, $w\mathbf{e}_1$ (where $\mathbf{e}_1$ is the first unit vector in $\mathbf{R}^m$) and 0 respectively. Then the static variational problem $(\mathcal{V}')$ becomes in the new notation:

$$\sup_{\tilde{c} \in \tilde{F}} E\left[\int_0^T \tilde{u}(t, \tilde{c}(t))\hat{\lambda}(\mathrm{d}t)\right] \tag{$\mathcal{V}'$}$$

where the set $\tilde{F}$ is given by the set of all $\tilde{c} \in (\mathcal{L}^0_+([0, T] \times \Omega))^m$ such that

$$\sup_{P^* \in \mathcal{P}} E^*\left[\int_0^T \gamma^0(t)\left(\tilde{s}(t) - \tilde{y}(t)\right)\hat{\lambda}(\mathrm{d}t) - A^{\mathcal{Y}}_{P^*}(T)\right] \leq w_0,$$

where $\tilde{s} = \sum_{i=1}^m \tilde{c}_i$. Note that for $\tilde{c} \in (\mathcal{L}^1(\hat{\lambda} \times P^{0\star}))^m$ the expectation in $(\mathcal{V}')$ is not $+\infty$, because of the asymptotic growth property (γ_1).

Before we start with the proof of the theorem, we first take a closer look at the set $\tilde{F}$:

Lemma 1 *The set $\tilde{F}$ is convex, subset of some ball in $(\mathcal{L}^1(\hat{\lambda} \times P^{0\star}))^m$ and closed with respect to convergence almost everywhere.*

Proof Let $\tilde{c} \in \tilde{F}$, then we deduce – recalling (2) and that $A^{\mathcal{Y}}_{P^{0\star}} \equiv 0$ since $P^{0\star} \in \mathcal{P}^0$–

$$E^{0\star}\left[\int_0^T \gamma^0(t)\tilde{s}(t)\hat{\lambda}(\mathrm{d}t)\right] \leq w_0 + E^{0\star}\left[\int_0^T \gamma^0(t)\tilde{y}(t)\hat{\lambda}(\mathrm{d}t)\right] \leq w_0 + K, \tag{11}$$

where $\tilde{s} = \sum_{i=1}^m \tilde{c}_i$. Thus, since $\int_0^T r(s)\mathrm{d}s < R$ by (1), $\tilde{F}$ is subset of some ball in $(\mathcal{L}^1(\hat{\lambda} \times P^{0\star}))^m$ The convexity of $\tilde{F}$ easily follows from the linearity of the inequalities. Finally, let $(\tilde{c}_n)_n$ be a sequence in F converging almost everywhere to, say, $\hat{c}$. By Fatou's lemma

$$\begin{aligned} E^*\left[\int_0^T \gamma^0(t)\sum_{i=1}^m (\hat{c}_n)_i(t)\hat{\lambda}(\mathrm{d}t)\right] &\leq \liminf_{n\to\infty} E^*\left[\int_0^T \gamma^0(t)\sum_{i=1}^m (\tilde{c}_n)_i(t)\hat{\lambda}(\mathrm{d}t)\right] \\ &\leq w_0 + E^*\left[\int_0^T \gamma^0(t)\tilde{y}(t)\hat{\lambda}(\mathrm{d}t) + A^{\mathcal{Y}}_{P^*}(T)\right], \end{aligned}$$

for all $P^* \in \mathcal{P}$. Thus $\hat{c}$ is in $\tilde{F}$. □

By the previous Lemma we see that the program $(\mathcal{V}')$ is an optimisation problem over the closed and norm-bounded subset $\tilde{F}$ of $(\mathcal{L}^1(\hat{\lambda} \times P^{0\star}))^m$. Since $\mathcal{L}^1$ is not reflexive, the set $\tilde{F}$ lacks weak compactness. To obtain existence, we will need the concept of *K-convergence* [18]. Let $(\Omega, \mathcal{S}, \mu)$ be a measure space, where Ω is a set and $\mathcal{S}$ and μ are a σ-algebra and a measure on Ω, respectively. Suppose $(f_n)_n$ is a sequence of integrable functions on this measure space. This sequence is said to K-converge to a function f if for every subsequence $(n') \subset (n)$, there exists a μ-null set $\tilde{N}$ such that, as $N \to \infty$,

$$\frac{1}{N}\sum_{n'=1}^{N} f_{n'}(\omega) \to f(\omega) \text{ for all } \omega \in \Omega\backslash\tilde{N}. \tag{12}$$

To obtain existence we use a deep result of Komlós [80]:

Theorem 3 (Komlós) *Let $(f_n)_n$ be functions satisfying $\sup_n \int_\Omega |f_n| d\mu < \infty$. Then there exists a μ-integrable f_* and a subsequence $(n') \subset (n)$ such that $f_{n'}$ K-converges to f_* as $n' \to \infty$.*

Proof of theorem 2 Define $s \equiv \sup(\mathcal{V}')$. Note that $s < \infty$ because of the growth property (γ_1). If $s = -\infty$ there is nothing to prove, so without loss of generality we can suppose $s \in \mathbf{R}$. Let $(\tilde{c}_n)_n \subset \tilde{F}$ be a maximising sequence for $(\mathcal{V}')$; that is, $E[\int_0^T \tilde{u}(t, \tilde{c}_n(t))\hat{\lambda}(\mathrm{d}t)] \uparrow s$ as $n \to \infty$. Since $\tilde{F}$ is subset of some ball in $(\mathcal{L}^1(\hat{\lambda} \times P^{0\star}))^m$, an application of Theorem 3 shows that there exists a subsequence $(n') \subset (n)$ for which $(\tilde{c}_{n'})_{n'}$ K-converges to a function $\tilde{c}_* : \Omega \times [0,T] \to \mathbf{R}^m$. By Lemma 1, $\tilde{F}$ is closed and convex; thus $\tilde{c}_* \in \tilde{F}$. By concavity of $\tilde{u}$, we find

$$\frac{1}{N}\sum_{n'=1}^{N} E\left[\int_0^T \tilde{u}(t, \tilde{c}_{n'}(t))\hat{\lambda}(\mathrm{d}t)\right] \leq E\left[\int_0^T \tilde{u}\left(t, \frac{1}{N}\sum_{n'=1}^{N}\tilde{c}_{n'}(t)\right)\hat{\lambda}(\mathrm{d}t)\right]. \tag{13}$$

As N approaches infinity, the left-hand side of (13) converges to s, where we used the fact that convergence of a sequence of real numbers implies convergence of the averages of the first l terms as l goes to infinity. To obtain convergence on the right-hand side of (13), we first decompose the expectation into two parts, using the fact that $\tilde{u} = \tilde{u}^+ - \tilde{u}^-$, where $\tilde{u}^+ = \max\{0, \tilde{u}\}$ and $\tilde{u}^- = -\max\{0, -\tilde{u}\}$. The expectation can be decomposed since $E\int \tilde{u}^+ < \infty$ by growth property (γ_1). Note that limsup $E\int(u^+ - u^-) \leq$ limsup $E\int u^+ -$ liminf $E\int u^-$. For the second part, Fatou's lemma combined with the continuity of u implies

$$\liminf_{N\to\infty} E\left[\int_0^T \tilde{u}^-\left(t, \frac{1}{N}\sum_{n'=1}^{N}\tilde{c}_{n'}(t)\right)\hat{\lambda}(\mathrm{d}t)\right] \geq E\left[\int_0^T \tilde{u}^-\left(t, \tilde{c}_*(t)\right)\hat{\lambda}(\mathrm{d}t)\right].$$

To deal with the first part we first note that the growth property (γ_1) implies that $\{\tilde{u}^+(t, s_r(t,\omega), \omega), r \in \mathbf{N}\}$, where $s_r = r^{-1}\sum_{n'=1}^{r} c_{n'}$, is uniformly integrable with respect to $\hat{\lambda} \times P$. Indeed, let $\epsilon > 0$, then by the growth property

(γ_1) there exists a $\tilde{\phi}^\epsilon \in \mathcal{L}^1(\hat{\lambda} \times P^{0\star})$ such that for each $A \in \mathcal{B}([0,T]) \times \mathcal{F}$

$$\begin{aligned}\int_A \tilde{u}^+(t, \tilde{s}_r(t,\omega), \omega)\mathrm{d}(\hat{\lambda} \times P) &\leq \epsilon \int_A |\tilde{s}_r(t,\omega)|\mathrm{d}(\hat{\lambda} \times P^{0\star}) \\ &\quad + \int_A \tilde{\phi}^\epsilon(t,\omega)\mathrm{d}(\hat{\lambda} \times P^{0\star}) \\ &\leq \epsilon \mathrm{e}^{-R}(w_0 + K) + \int_A \tilde{\phi}^\epsilon(t,\omega)\mathrm{d}(\hat{\lambda} \times P^{0\star}).\end{aligned}$$

By first choosing ϵ small enough and then choosing A, this quantity can be made arbitrarily small, uniformly in r. Since almost everywhere-convergence together with uniform integrability implies convergence in $\mathcal{L}^1$ and $u(t,\cdot,\omega)$ is continuous, we find that

$$\limsup_{N\to\infty} E\left[\int_0^T \tilde{u}^+\left(t, \frac{1}{N}\sum_{n'=1}^N \tilde{c}_{n'}(t)\right)\hat{\lambda}(\mathrm{d}t)\right] = E\left[\int_0^T \tilde{u}^+(t, \tilde{c}_*(t))\hat{\lambda}(\mathrm{d}t)\right].$$

Thus, $s \leq E[\int_0^T \tilde{u}(t, \tilde{c}_*(t,\omega), \omega)\hat{\lambda}(\mathrm{d}t)]$ and $\tilde{c}_*$ is an optimal solution for $(\mathcal{V}')$. Combining with Theorem 1, we immediately find an existence result for the original problem $(\mathcal{V})$. □

5 Characterisation of optimal policies

In the previous section, Theorem 2 gives conditions under which the existence of an optimal consumption-final wealth plan (c_*, w_*) is guaranteed, but it tells nothing about the actual form of (c_*, w_*). In this section we assume that the utility functions u, v satisfy the conditions of Theorem 2. Denote by $\mathbf{e}$ the column vector of m ones and let $(c_*(t,\omega))_{\mathrm{diag}}$ be the diagonal matrix with $c_{*i}(t,\omega)$, $i = 1,\ldots,m$ on its diagonal. Furthermore, by $B_\epsilon(c_*, w_*)$ we denote the set of all $(c,w) \in \mathcal{A}^0(\mathcal{H})$ that satisfy

$$\max\{|c_1 - c_{*1}|, \ldots, |c_m - c_{*m}|, |w - w_*|\} \leq \epsilon \max\{|c_{*1}|, \ldots, |c_{*m}|, |w_*|\}$$

for almost every (t,ω). The column vector of coordinate-wise absolute values of the gradient of u, $(|\frac{\partial u}{\partial c_1}|, \ldots, |\frac{\partial u}{\partial c_m}|)^\top$, we will denote for short by $|\nabla_c| u$. Then we have the following result, which relates the marginal utility at the optimal consumption-final wealth plan (c_*, w_*) with Radon-Nikodym derivatives of $P^* \in \mathcal{P}$ with respect to P. A similar result in the context of one-dimensional intertemporal consumption and where stock prices are modelled as Itô-processes can be found in Cuoco [45].

Proposition 2 *Suppose that $\mathcal{H}$ is such that the set $\mathcal{P}$ is convex, that $(c_*, w_*) \neq 0$ and that there exists a $\epsilon_0 \in (0,1)$ such that (u,v) is differentiable for $(c,w) \in B_{\epsilon_0}(c_*, w_*)$ and*

$$E\left[\int_0^T c_*(t)^\top |\nabla_c|\, u(t, c(t))\mathrm{d}t + w_* \left|\frac{\partial v}{\partial w}(w)\right|\right] < \infty. \tag{14}$$

for all $(c,w) \in B_{\epsilon_0}(c_*, w_*)$. *Let* (c_*, w_*) *be an optimal solution of* $(\mathcal{V})$ *Then there exists a sequence* $(\phi_n \xi_n)_n$ *with* $\phi_n > 0$ *and* ξ_n *the Radon-Nikodym derivative of* $P_n^* \in \mathcal{P}$ *with respect to* P *such that*

$$\begin{aligned}(c_*(t,\omega))_{\mathrm{diag}}(\nabla_c u(t, c_*(t,\omega),\omega) - \gamma^0(t,\omega)\phi_n\xi_n(t,\omega)\mathbf{e}) \to 0 \\ w_*(\omega)(\tfrac{\partial v}{\partial w}(w_*(\omega),\omega) - \gamma^0(T,\omega)\phi_n\xi_n(T,\omega)) \to 0\end{aligned} \tag{15}$$

in $\left(L^1(\lambda \times P)\right)^m$ *and* $L^1(P)$ *respectively as* $n \to \infty$. *If in addition*

$$\inf_{P^* \in \mathcal{P}} E^* \left[\int_0^T \gamma^0(t) c_*(t)^\top \mathbf{e} \mathrm{d}t + \gamma^0(T) w_*\right] > 0, \tag{16}$$

then (15) holds with $\phi_n = \phi > 0$ *for all* n.

Recall from Example 2 that the set $\mathcal{P}$ is convex for example if $\mathcal{H}$ is a linear family or a convex cone. If the optimal consumption c_* and the final wealth w_* are uniformly bounded away from zero, one has as an immediate consequence from above proposition that the marginal utility of inter-temporal consumption and that of final wealth at the optimal consumption-final wealth plan (c_*, w_*) are, up to a constant multiplicative factor, equal to a pointwise limit of 'state price densities' $\gamma_0\xi_n$:

Corollary 1 *Assume that (14) and (16) hold and* $\mathcal{H}$ *is such that the set* $\mathcal{P}$ *is convex. Then there exist a* $\phi > 0$ *and a sequence* $(\xi_n)_n$ *with* $\xi_n(t) = E[\frac{\mathrm{d}P_n^*}{\mathrm{d}P}|\mathcal{F}_t]$ *for* $P_n^* \in \mathcal{P}$ *such that if* $c_{*i} > 0$ *a.e.*

$$(\nabla_c u(t, c_*(t,\omega),\omega))_i = \lim_{n\to\infty} \gamma^0(t,\omega)\phi\xi_n(t,\omega) \qquad \lambda \times P\text{-a.e.} \tag{17}$$

and if $w_* > 0$ *a.e.*

$$\frac{\partial v}{\partial w}(w_*(\omega),\omega) = \lim_{n\to\infty} \gamma^0(T,\omega)\phi\xi_n(\omega) \qquad P\text{-a.e.} \tag{18}$$

Proof of Proposition 2 Recall the notations $\tilde{u}$ and $\tilde{c}$ from Section 4.1 and write $\nabla_c \tilde{u}(s)$ for $\nabla_c \tilde{u}(s,c)|_{c=\tilde{c}_*(s)}$, the gradient of $\tilde{u}(s,c)$ in $c = \tilde{c}_*(s)$. Inspired by [45] we define the sets C_1 and C_2, which are subset of $(L^1(\hat{\lambda} \times P))^m$ by (8), 2) and Assumption 2), by

$$C_1 = \{\phi\gamma_0\xi_{P^*} \times \tilde{c}_* : \phi > 0, P^* \in \mathcal{P}\}$$

where $\xi_{P^*}(t) = E[\frac{\mathrm{d}P^*}{\mathrm{d}P}|\mathcal{F}_t]$ and

$$C_2 = \{(\tilde{c}_*)_{\mathrm{diag}}\nabla_c\tilde{u} - x : x \in \mathrm{cl}(C_1)\}.$$

where $\mathrm{cl}(C_1)$ denotes the closure of C_1 in $(L^1(\hat{\lambda} \times P))^m$. Since $\mathcal{P}$ is assumed to be convex, C_2 is convex as well. We argue by contradiction and suppose that there is no sequence $(\phi_n\xi_n)_n$ such that $\phi_n\gamma^0\xi_n\tilde{c}_* \to \tilde{u}(\tilde{c}_*)_{\mathrm{diag}}\nabla_c$. Then

$C_2 \cap \{0\} = \emptyset$ and it follows therefore from the separating hyperplane theorem (e.g. [51, Thm. V.2.10]) that there exists an $f \in (L^\infty(\hat\lambda \times P))^m$ such that

$$E\left[\nabla_c \tilde u(s)^\top (\tilde c_*(s))_{\mathrm{diag}} f(s) \hat\lambda(\mathrm{d}s)\right] - \phi E^*\left[\int_0^T \gamma^0(s)\mathbf{e}^\top (\tilde c_*(s))_{\mathrm{diag}} f(s)\hat\lambda(\mathrm{d}s)\right] > 0$$

for all $P^* \in \mathcal{P}$ and $\phi > 0$. Writing $\hat f = (\tilde c_*)_{\mathrm{diag}} f / \|f\|_{L^\infty}$, the above implies

$$E\left[\int_0^T \nabla_c \tilde u(s)^\top \hat f(s)\hat\lambda(\mathrm{d}s)\right] > 0 \geq E^*\left[\int_0^T \gamma^0(s)\mathbf{e}^\top \hat f(s)\hat\lambda(\mathrm{d}s)\right], \tag{19}$$

for all $P^* \in \mathcal{P}$. Note that for $\epsilon \in (0,1)$ the consumption plan $\tilde c_\epsilon = \tilde c_* + \epsilon \hat f$ satisfies $\tilde c_\epsilon \geq (1-\epsilon)\tilde c_* \geq 0$. But then by Theorem 1 the consumption-final wealth plan $(\tilde c_\epsilon, \tilde c_\epsilon(T))$ is $\mathcal{H}$-feasible for each $\epsilon \in (0,1)$. It follows from the optimality of $\tilde c_*$ that

$$0 \geq \lim_{\epsilon \downarrow 0} \frac{U(c_\epsilon, w_\epsilon) - U(c_*, w_*)}{\epsilon} = E\left[\int_0^T \nabla_c \tilde u(s, \tilde c_*(s))\hat f(s)\hat\lambda(\mathrm{d}s)\right],$$

where the last equality follows from the dominated convergence theorem using the fact that by concavity of u for $\epsilon > 0$ small enough

$$\begin{aligned}\frac{\tilde u(t, \tilde c_\epsilon(t)) - \tilde u(t, \tilde c_*(t))}{\epsilon} &\leq \max\{|\nabla_c \tilde u(t, \tilde c_{\epsilon_0}(t))^\top \hat f(t)|, |\nabla_c \tilde u(t, \tilde c_*(t))^\top \hat f(t)|\} \\ &\leq (|\nabla_c|\tilde u(t, \tilde c_{\epsilon_0}(t)) + |\nabla_c|\tilde u(t, \tilde c_*(t)))^\top \tilde c_*(t)\end{aligned}$$

and that the last expression is integrable by (14). This contradicts (19) and thus proves (15).

Now suppose that in addition condition (16) holds and let $(\varphi_n \xi_n)_n$ be a sequence as in (15). Since $\varphi_n \|\gamma^0 \xi_n \mathbf{e}^\top \tilde c_*\|_{L^1}$ converges to $\|(\tilde c_*)^\top \nabla_c \tilde u\|_{L^1}$ (where $\|\cdot\|_{L^1}$ denotes the $L^1(\hat\lambda \times P)^m$-norm) and $\|\gamma^0 \xi_n \mathbf{e}^\top (\tilde c_*)\|_{L^1}$ is bounded away from zero, $(\varphi_n)_n$ is bounded. Hence we can find a subsequence, again denoted by $(\varphi_n)_n$, such that $\varphi_n \to \varphi > 0$. Then by the triangle inequality and (11), $\|\varphi \gamma_0 \xi_n \mathbf{e}^\top \tilde c_* - (\tilde c_*)^\top \nabla_c \tilde u\|_{L^1}$ is bounded by

$$\begin{aligned}&\|\gamma_0 \xi_n \mathbf{e}^\top \tilde c_*\|_{L^1}|\varphi - \varphi_n| + \|\varphi_n \gamma_0 \xi_n \mathbf{e}^\top \tilde c_* - (\tilde c_*)^\top \nabla_c \tilde u\|_{L^1} \\ &\qquad \leq (w_0 + K)|\varphi - \varphi_n| - \|\varphi_n \gamma_0 \xi_n \mathbf{e}^\top \tilde c_* - (\tilde c_*)^\top \nabla_c \tilde u \cdot \|_{L^1},\end{aligned}$$

which tends to zero as $n \to \infty$. The proof is done. □

Remark. Suppose we are in the setting of Corollary 1 and suppose that $v(w,\omega)$ is differentiable and strictly concave in w with $\partial_w v(w,\omega)$ tending to zero and ∞ if $w \to \infty$ and $w \downarrow 0$ respectively. Then $\frac{\partial v}{\partial w}(\cdot,\omega) : [0,\infty] \to [0,\infty]$ is strictly decreasing and we denote its inverse function by $I(\cdot,\omega)$. Writing then ξ_* for the pointwise limit of the ξ_n from (18) we find for the optimal final wealth w_*

$$w_*(\omega) = I(\phi\gamma_0(T,\omega)\xi_*(\omega), \omega). \tag{20}$$

Note that under appropriate conditions on u, a similar result holds for c_{*i}. If ξ_* is a Radon-Nikodym derivative of some probability measure with respect to P, the formula (20) is reminiscent of the results found by Kramkov and Schachermayer [75]. The exact relation between the direct approach and the dual one as developed by [75, 76] is subject of ongoing research.

6 Examples

6.1 A Jump-diffusion model

In this section we consider as a specific model for the price of the risky assets a jump-diffusion driven by a Wiener process and an independent Poisson process.

Assume that on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$ there exists a d-dimensional Wiener process $W = (W^{(1)}, \ldots, W^{(d)})$ with independent components ($d \geq n$) and an s-dimensional Poisson processes $N = (N^{(1)}, \ldots, N^{(s)})$ with $\mathbf{F}$-adapted intensity ν and independent components. The price process of the ith risky assets $S_i = \{S_i(t), 0 \leq t \leq T\}$ is modelled by the stochastic differential equation

$$\mathrm{d}S_i(t) = S_i(t)\left[\mu_i(t)\mathrm{d}t + \sigma_i(t)^\top \mathrm{d}W(t) + \rho_i(t)^\top \mathrm{d}N(t)\right].$$

Here μ and σ are $\mathbf{F}$-adapted and ρ is locally bounded from below and $\mathbf{F}$-predictable and they satisfy

$$\int_0^T |\mu(s)|\mathrm{d}s + \int_0^T |\sigma(s)|^2\mathrm{d}s + \int_0^T |\rho(s)\cdot\nu(s)|\mathrm{d}s + \int_0^T |\nu(s)|\mathrm{d}s < \infty.$$

Moreover, we assume that the $n \times d$ matrix σ with rows σ_i has full rank n. Note that this market is incomplete if $d > n$ or the intensity ν is nonzero. The market is free of arbitrage if there exists a process $\kappa = (\kappa_i)_{i=1}^s$ which solves

$$\sigma(t)\kappa(t) = \mu(t) + \rho(t)\nu(t) - r(t)\mathbf{e}, \qquad t \in [0,T],$$

where $\mathbf{e} \in \mathbf{R}^n$ is the column-vector of ones and satisfies the so called *Novikov condition* condition

$$E\left[\exp\left\{2^{-1}\int_0^T |\kappa(s)|^2\mathrm{d}s\right\}\right] < \infty. \tag{21}$$

Indeed, applying Itô's lemma to the process $Z = \{Z_t : t \in [0,T]\}$ with

$$Z(t) = \exp\left\{\int_0^t \kappa(s)\mathrm{d}W(s) - \frac{1}{2}\int_0^t |\kappa(s)|^2\mathrm{d}s\right\}$$

shows that Z satisfies the stochastic differential equation $\mathrm{d}Z = Z\kappa\mathrm{d}W$ and is thus a local martingale. It is well known that under (21) (e.g. [71]) Z becomes a martingale. Moreover, by Girsanov's theorem the discounted price process $\gamma^0 S$ is a local martingale under the measure P^0 given by $\mathrm{d}P^0 = Z\mathrm{d}P$.

6.2 Complete markets

In this section we consider the case that the set of equivalent local martingale measures is a singleton, $\mathcal{P}^0 = \{P^0\}$ and no constraints are put on the trade, that is, the portfolios may take values in $\mathcal{H} = \mathbf{L}^a_{loc}(X)$. In this setting the market is arbitrage free (cf. Proposition 1) and complete.

Now we are also interested in the integrability of the consumption final-wealth plans (c, w). For $p \geq 1$ we define $\mathcal{A}^p$ as the subset of $\mathcal{A}^0 = \mathcal{A}^0(\mathbf{L}^a_{loc}(X))$ consisting of the p-integrable consumption-final wealth plans

$$\mathcal{A}^p \equiv \{(c, w) \in \mathcal{A}^0 : (c, w) \in (\mathcal{L}^p_+(\lambda \times P))^m \times \mathcal{L}^p_+(P)\}. \tag{22}$$

where $\mathcal{L}^p_+(\cdot)$ denotes the set of non-negative p-integrable functions with respect to the measure $\cdot$ between the brackets. In this setting, the agent faces the following problem for $p = 0$ or $p \geq 1$

$$\sup_{(c,w)\in\mathcal{A}^p} E\left[\int_0^T u(t, c(t, \omega), \omega)\mathrm{d}t + v(w(\omega), \omega)\right] \tag{$\mathcal{V}_p$}$$

By Theorem 1 we now can reformulate the dynamic control problem $(\mathcal{V}_p)$ as the following static variational problem. As before, we denote by ξ the Radon-Nikodym derivative of P^0 with respect to P, $\xi(t) = \xi^0(t) = E\left[\mathrm{d}P^0/\mathrm{d}P|\mathcal{F}_t\right]$.

$$\sup_{(c,w)} E\left[\int_0^T u\left(t, c(t)\right)\mathrm{d}t + v(w)\right] \text{ s.t. } E\left[\int_0^T \pi(t)z(t)\mathrm{d}t + \pi(T)w\right] \leq w_0. \tag{$\mathcal{V}'_p$}$$

where $\pi(t) = \gamma^0(t)\xi(t)$ and $z = \sum_{i=1}^m c_i - y$ and the supremum is taken over the set of (c, w) in $(\mathcal{L}^p_+(\lambda \times P))^m \times \mathcal{L}^p_+(P)$.

Existence

Note $(\mathcal{V}'_p)$ is an optimisation problem of the form studied by [19, 44]. As the following examples show, $(\mathcal{V}'_p)$ may have no solution.

Example 4 *Let the consumption plans be one-dimensional ($m = 1$) and assume that $\xi^{-1} \in \mathcal{L}^p_+(\lambda \times P)$. Set $u(t, c)$ equal to $c \times t$ whereas $v \equiv 0$. Consider then the sequence $(c_n, w_n)_n$ of consumption-final wealth plans given by $w_n = 0$ and*

$$c_n(t) = nk\mathbf{1}_{[T-\frac{1}{n},T]}(t) \cdot S^0(t)\xi^{-1}(t) \qquad n = 1, 2, \ldots$$

where $k = w_0 + E[\int_0^T \pi(t)y(t)dt]$. Since $\xi^{-p} \in \mathcal{L}^1_+(\lambda \times P)$, we see that $c_n \in \mathcal{L}^p_+(\lambda\times P)$ and that (c_n, w_n) satisfies the constraint in $(\mathcal{V}'_p)$ as equality. Hence the (c_n, w_n) are feasible for $(\mathcal{V}'_p)$. However, $E[\int_0^T u(t, c_n(t))dt] \to \infty$ as $n \to \infty$. In this case $(\mathcal{V}'_p)$ has no solution, since the agent would like to concentrate his/her consumption closer and closer to time $t = T$. ◇

Example 5 *We show that the supremum in* $(\mathcal{V}'_p)$ *may not be attained, although it is finite. We assume that* $(\lambda \times P)(\pi^0_t \in (1, 1+\epsilon)) > 0$ *for some* $\epsilon > 0$. *Consider* $(\mathcal{V}'_p)$ *in the same setting as in the previous example, but now with* $u(c,t) = c \cdot \mathbf{1}_A(t,\omega)$, *where* $A \equiv \{(t,\omega) : \pi^0(t,\omega) > 1\}$. *Note*

$$E\left[\int_0^T \mathbf{1}_A(t)c(t)\,dt\right] < E\left[\int_0^T \xi(t)\gamma^0(t)c(t)\,dt\right] \leq k \tag{23}$$

where k *is as in the previous example. From (23) we see that the supremum in* $(\mathcal{V}'_p)$ *is at most* k. *The sequence given by*

$$c_n(t,\omega) = \frac{k}{T}\frac{\mathbf{1}_{G_n}(t,\omega)}{E\left[\int_0^T \gamma^0(t)\xi(t)\mathbf{1}_{G_n}(t)\,dt\right]},$$

where $G_n = \{(t,\omega) : 1 < \gamma^0(t,\omega)\xi(t,\omega) < 1 + \frac{1}{n}\}$ *shows that in fact the supremum is* k. *However, since the first inequality in (23) is a strict one, a feasible consumption* c *plan with expected utility* $E\left[\int_0^T u(t, c(t))\,dt\right]$ *equal to* k *does not exist.* $\diamond$

From the above examples, it appears, certain conditions on the asymptotic growth rate of u and v are needed to ensure existence in $(\mathcal{V}'_p)$. Therefore, following [19], we introduce the following asymptotic growth condition $(\widetilde{\gamma}_p)$ on u and v (closely connected to the condition (γ_1)):

$$\left\{\begin{array}{c} \text{For every } \epsilon > 0 \text{ there exist a } \psi^\epsilon \in \mathcal{L}^p(P) \text{ and a } \phi^\epsilon \in \mathcal{L}^p(\lambda \times P), \\ \text{such that for all } c \in \mathbf{R}^m_+ \text{ and } w \in \mathbf{R}_+ \\ v(w,\omega) \leq \epsilon\xi(T,\omega)w + \xi(T,\omega)\psi^\epsilon(\omega) \text{ for } P\text{-a.s. } \omega \in \Omega \\ u(t,c,\omega) \leq \epsilon\xi(t,\omega)|c| + \xi(t,\omega)\phi^\epsilon(t,\omega) \text{ for } \lambda \times P\text{-a.s. } (t,\omega) \in [0,T] \times \Omega \end{array}\right\} \tag{$\widetilde{\gamma}_p$}$$

In addition, it appears, that, in order to guarantee existence in $(\mathcal{V}'_p)$, u and v have to satisfy the condition of *essential non-satiation* (ς); i.e. there exist sets I, J with $P(I) > 0$ and $(\lambda \times P)(J) > 0$ such that

$$\left\{\begin{array}{ll} \arg\max_{w \in \mathbf{R}_+} v(w,\omega) = \emptyset & \text{for all } \omega \in I \\ \arg\max_{c \in \mathbf{R}^m_+} u(t,c,\omega) = \emptyset & \text{for all } (t,\omega) \in J \end{array}\right\} \tag{ς}$$

Now we can state the existence result for $(\mathcal{V}_p)$, $p \geq 1$, which follows immediately from combining [19, Theorem 2.8] and [19, §5.2] with Theorem 1:

Theorem 4 *Suppose that* $u(t,z,\omega)$ *and* $v(z,\omega)$ *are upper semi-continuous in* z *for a.e.* (t,ω) *in* $[0,T] \times \Omega$ *and a.e.* ω *in* Ω *respectively, that* $v(z,\omega)$ *is concave in* z *for a.e.* ω *in the purely atomic part* Ω^{pa} *of* $(\Omega, \mathbf{F}, P)$ *and that* u *and* v *are essentially nonsatiated* (ς). *Suppose also that* u, v *satisfy growth condition* $(\widetilde{\gamma}_p)$ *and that there exists some* $(\tilde{c}, \tilde{w}) \in \mathcal{L}^p_+(\lambda^m \times P) \times \mathcal{L}^p(P)$ *for which* $(t,\omega) \mapsto u(t, \tilde{c}(t,\omega), \omega)/\xi(t,\omega)$ *belongs to* $\mathcal{L}^p_+(\lambda \times P)$ *and* $\omega \mapsto v(\tilde{w}(\omega), \omega)/\xi(T,\omega)$ *belongs to* $\mathcal{L}^p(P)$. *Then problem* $(\mathcal{V}_p)$ *has an optimal solution.*

For concrete examples of utility functions satisfying the requirements of Theorem 4 we refer to Example 3, where for utility functions of intermediate consumption one replaces the requirement of concavity by upper semicontinuity. For example the utility function

$$u(t,c,\omega) = (1-\exp(-|c|^2/2\sigma^2))\mathbf{1}_{F_t}(\omega) \qquad c \in \mathbf{R}_+^m, \sigma \neq 0,$$

where $F_t \in \mathcal{F}_t$ with $P(F_t) > 0$, satisfies the requirements of Theorem 4 but fails to satisfy those of Theorem 2, since it is not concave.

Characterisation of optimal consumption-final wealth plan

In this subsection, we look at characterisation of the optimal solutions of $(\mathcal{V}'_p)$ for $p = 0$ or $p \geq 1$. If $w_0 + E[\int_0^T \pi(t)y(t)\mathrm{d}t] = 0$, by Proposition 1, we only have $(c,w) = 0$ a.e. as admissible consumption-final wealth plan for $(\mathcal{V}'_p)$. So, let us assume $w_0 + E[\int_0^T \pi(t)y(t)\mathrm{d}t] > 0$. Moreover, we suppose $v(\cdot,\omega)$ is concave for P-a.e. ω in the purely atomic part Ω^{pa} of Ω. Then, from e.g. [1], we find that, (c_*, w_*) is optimal for $(\mathcal{V}'_p)$ if and only if (c_*, w_*) is a feasible consumption-final wealth plan for $(\mathcal{V}'_p)$ and there exists a $\zeta \geq 0$ such that the following two conditions hold:

$$\zeta\left(E\left[\int_0^T \pi(t)z(t)\mathrm{d}t + \pi(T)w\right] - w_0\right) = 0 \tag{CS}$$

$$\begin{cases} c_*(t,\omega) \in \mathrm{argmax}_{x\in\mathbf{R}_+^m} u(t,x,\omega) - \zeta x^\top \mathbf{e}\pi(t,\omega) & \lambda \times P\text{-a.e.} \\ w_*(\omega) \in \mathrm{argmax}_{x\in\mathbf{R}_+} v(x,\omega) - \zeta x\pi(T,\omega) & P\text{-a.e.,} \end{cases} \tag{PMP}$$

where, as before, we wrote $z = \sum_i c_i - y$ and $\pi = \gamma^0\xi$. The foregoing two equations are also known as complementary slackness (CS) and the pointwise maximum principle (PMP) respectively. Note no concavity of u and v is demanded, except for v on Ω^{pa}. If, in addition, u, v satisfy the condition of essential non-satiation (ς), $\zeta > 0$ in the above characterisation. Then the condition of complementary slackness is equivalent to

$$E\left[\int_0^T \left(\pi(t)\sum_{i=1}^m c_{*i}(t)\right)\mathrm{d}t + \pi(T)w_*\right] = w_0 + E\left[\int_0^T \pi(t)y(t)\mathrm{d}t\right] \tag{24}$$

Example 6 *Consider the problem $(\mathcal{V}'_p)$ for $p = 0$ or $p \geq 1$, where the agent faces a consumption-investment problem, where just one commodity is available and the utility of final wealth equal to zero. Suppose that for almost every $(\omega,t) \in \Omega \times [0,T]$ the utility function $u(t,\cdot,\omega)$ is differentiable, increasing and strictly concave on $\mathbf{R}_+$, with $u'(t,0,\omega) \equiv \lim_{z\downarrow 0}(\partial_z u)(t,z,\omega) = +\infty$ and $u'(t,\infty,\omega) \equiv \lim_{z\uparrow\infty}(\partial_z u)(t,z,\omega) = 0$. Note that in (24) the optimal w_* is zero a.e., by monotonicity of $u(t,\cdot,\omega)$. The above then implies that c_* is optimal in $(\mathcal{V}'_p)$ if and only if there exists a $\zeta > 0$ such that (24) and the pointwise maximum* *principle (PMP) are satisfied for a.e.* (t,ω) *in* $[0,T]\times\Omega$. *By differentiability and concavity of* $u(t,\cdot,\omega)$ *and since* $u'(t,0,\omega)=+\infty$, *(PMP) is equivalent to*

$$\partial_z u(t,z,\omega)\big|_{z=c_*(t,\omega)} = \zeta\pi(t,\omega) \text{ for } P\times\lambda\text{-a.e. } (\omega,t)\in\Omega\times[0,T]. \tag{25}$$

For all (t,ω) *for which* $u(t,\cdot,\omega)$ *is strictly concave, it has a (strictly) decreasing derivative and* $\partial_z u(t,\cdot,\omega)$ *has an inverse* $\mathcal{I}(t,\cdot,\omega)\equiv(\partial_z u)^{-1}(t,\cdot,\omega):[0,\infty]\to[0,\infty]$. *Thus we can equivalently rewrite (25) as*

$$c_*(t,\omega)=\mathcal{I}(t,\pi(t,\omega),\omega) \text{ for } P\times\lambda\text{-a.e. } (\omega,t)\in\Omega\times[0,T].$$

Now we introduce the function $\mathcal{J}$ *on* $\mathbf{R}_+$ *by*

$$\mathcal{J}(y)=E\left[\int_0^T \pi(t,\omega)\mathcal{I}(y\pi(t,\omega),t,\omega)\,dt\right]$$

and assume that $\mathcal{J}(y)<\infty$ *for all* $y\in(0,\infty)$. *By the monotone convergence theorem and the dominated convergence theorem* $\mathcal{J}$ *is (strictly) decreasing and continuous. Moreover, we find* $\lim_{y\downarrow 0}\mathcal{J}(y)=+\infty$ *and* $\mathcal{J}(\infty)=0$. *Thus* $\mathcal{H}$ *has an inverse* $\mathcal{K}\equiv\mathcal{J}^{-1}$ *and there is an unique* $\zeta=\mathcal{K}(w_0+E[\int_0^T\pi(t)y(t)\,dt])$ *satisfying (24). Hence, under the assumption that* $\mathcal{J}<\infty$ *on* $\mathbf{R}_+$, *the optimal consumption plan in* $(\mathcal{V}_0')$ *is given by*

$$c_*(t,\omega)=\mathcal{I}\left(t,\mathcal{K}\left(w_0+E\left[\int_0^T\pi(t)y(t)\,dt\right],\omega\right)\pi(t)\right).$$

If, for $p\geq 1$, c_* *is in addition* p*-integrable,* c_* *is the optimal consumption in* $(\mathcal{V}_p')$ *as well.* ◇

Acknowledgement. The author would like to thank Erik Balder for drawing the paper of Cuoco [45] to his attention and suggesting to generalise it.