Introduction

Risk plays a role in everyone's daily life. Some people are prepared to take more risk for example to achieve some goal, other people are quite 'risk-averse' and prefer to play safe. In financial markets similar patterns turn up with 'risk-loving' and 'risk-averse' agents. There risk can be *traded*: If an agent considers his financial position as too risky, he may limit the risk he is exposed to by buying an appropriate *contingent claim*. Such a claim is a contract between buyer and seller, where the latter promises to pay the former some payment or series of payments in the future. Contingent refers to the fact that at the time of agreement of the contract the actual size of the payment can be uncertain: The payment often depends on future developments (such as the price level of a certain stock one year from now). An example of a contingent claim is a *put (call) option*, which gives the right to sell (buy) a certain asset at a specified price until or at a future date.

History

The valuation of contingent claims is one of the main issues studied in modern finance: What is a fair price of a particular contingent claim? In other words, how much should the buyer of the claim pay to the seller such that both parties are satisfied (e.g. no of two parties can achieve a riskless profit)?

If the contract specifies that the holder has the right to exercise at a given future date, then this contingent claim is called a *European* option. In the literature, the pricing of European options goes back as early as Bachelier [17]. In 1900 he was the first to use Brownian motion with drift to model stock price fluctuations. In 1973 the papers of Black and Scholes [27] and Merton [97] appeared and would turn out to be milestones in the field: they established the important notions of *hedging* and *arbitrage free pricing*, which are currently common knowledge of traders worldwide. Harrison, Kreps and Pliska [66, 67] extended their ideas and put them on a firm mathematical basis using stochastic calculus. Almost 25 years later, in 1997, the Nobel Prize in Economics was awarded to Merton and Scholes for their path-breaking work (Black, who died in 1995, would undoubtedly have shared in the prize, had he still been alive.)

An option that turns up in practice more often, however, is the one where the holder has the right to exercise his contract at any time prior to the given future date. Claims of this type are called *American* options and their feature of intermediate exercise causes their valuation to be more complex and mathematically challenging. In this case, the question of the value of the option is intimately connected to that of the optimal exercise time of the holder. In 1965 McKean [99] was the first to give an analysis on pricing American options. He transformed the problem of pricing the American put into a *Stefan* or *free boundary* problem for the heat equation and solved this up to the free boundary. This free boundary corresponds to the *optimal exercise boundary*: It is optimal to exercise the put the first time that the stock price hits or falls below this space-time curve. Since then, a significant volume of literature has appeared on different aspects of pricing American options. See Myneni [103] for a review of the theory and methods of pricing American type options.

Modelling the stock price

The continuous time models we discussed until now all used the geometric Brownian motion as model for the evolution of the stock price. However, from extensive empirical research it appeared that this model is not ideal: It is not capable of replicating some of the features commonly seen in financial data, such as heavy tails and asymmetry. Recently, there has been a lot of interest in replacing the geometric Brownian motion by an exponential Lévy model which performs better empirically. A Lévy process is a stochastic process with stationary independent increments, whose paths are right-continuous and have left limits. The class of Lévy processes has a quite rich structure as is also demonstrated by the fact that it is in one-to-one correspondence with the class of infinitely divisible distributions. It is this flexibility that makes Lévy processes suitable for many modelling purposes. As most recent examples of stock price models driven by Lévy processes we mention the normal inverse Gaussian model proposed by Barndorff-Nielsen [20], the hyperbolic model of Eberlein [52], the variance-gamma model first explored by Madan and Seneta [92, 91] and the truncated stable family introduced by Koponen [32, 42, 81].

Replacing the standard geometric Brownian motion as model for the stock by an exponential Lévy process generally leads to several problems of different nature in answering the questions of valuation of contingent claims. In a market where the stock prices are driven by Brownian motion, the market is *complete*, that is, for every claim there exists a self-financing trading strategy such that the corresponding portfolio *replicates* the claim. By arbitrage arguments it then follows that the fair, arbitrage-free price of such a claim is equal to the initial value of its corresponding hedging strategy. Moreover, it turns out that the price can be evaluated as the expectation of the discounted claim under a *(local) martingale measure* equivalent to the 'real word' or 'objective' measure. This is a measure, also called a *risk neutral measure*, under which the discounted stock price becomes a local martingale. In a complete arbitrage free market an equivalent martingale measure exists and is unique.

Introducing jumps, however, generally leads to an *incomplete* market model. That is, in this market not all claims that can necessarily be replicated by a self-financing portfolio and if the market is free of arbitrage there exist infinitely many equivalent (local) martingale measures. It is therefore not clear what the fair price of these claims should be. Since a non-attainable claim can not be completely hedged against, for a particular agent the fair price of the claim will depend on his/her attitude towards risk. A possible approach is therefore to consider the pricing problem in the context of utility theory and link the pricing problem with utility optimisation problems of the agent.

A different approach of pricing in incomplete markets is based on selecting a particular local martingale measure as pricing measure. In analogy with the complete setting the price of the claim is then computed under this measure. In the literature different selection criteria have been developed, such as entropy minimisation and Esscher transformation, although it seems that the final word about this issue has not yet been spoken. For a review of the literature we refer to Chan [38] and references therein.

Whichever of the two approaches is taken, replacing the geometric Brownian motion model by an exponential Lévy process leads to many *mathematical* issues which need to be resolved to completely settle the problem of pricing options. The presence of jumps asks for adaptations of much of the previously mentioned theory connected to the classic geometric Brownian motion model. For example, in this model the value function of a European option with payoff only depending on the final value of the stock satisfies a partial differential equation. The possibility of jumps of the price process, however, introduces non-locality in the operator and, in the second approach mentioned above, we are led to the study of pseudo-differential equations. See e.g. [32] for recent work in this direction.

Organisation and outline of this thesis

The rest of this thesis consists of five self-contained chapters, each with its own summary and introduction, followed by a list of references. We give now an outline of the contents.

In the first chapter we study four options of American type in the context of the geometric Brownian motion model: the American put and call, the Russian option and the integral option. The value of the last two options was earlier computed in the papers [83, 116, 117]. We give an alternative derivation of their value exploiting properties of Brownian motion and Bessel processes. The four options we consider are all options of perpetual type, that is, they never expire. From a practical point of view perpetual options do not seem of much use, since in practice the time of expiration is always finite. However, following an appealing idea of Carr [36], one can build an approximating sequence of *perpetual-type* options that converges pointwise to the value of the corresponding finite time American option. This approximation procedure is also called *Canadization*. In Carr [36] numerical evidence was given for this convergence, here we give a mathematical proof. Next we compute for the three mentioned options the first approximation.

The second chapter proposes the *phase type* Lévy processes as a new model for the stock price. These are jump-diffusions whose positive and negative jumps form compound Poisson processes with jump distributions of *phase type*. Phase type distributions have a rational Laplace transform and are dense in all distributions. As a consequence, phase type Lévy processes form a class that is dense in all Lévy processes. Apart from this flexibility in modelling, the main reason for coming up with this new model is the analytical tractability of the pricing of many options under this model. We illustrate this by solving the problem of pricing the perpetual American put and Russian option under the phase type Lévy model. For the valuation we followed the second approach as sketched above, choosing as martingale measure the Esscher transform.

In the third chapter we study the same problems but now for the class of Lévy processes without negative jumps. We restrict ourselves to this class, since it contains already a lot of the rich structure of Lévy processes while still being analytically tractable due to many available results exploiting the fact that the jumps of the Lévy process have one sign. A recent study [37] offers empirical evidence supporting the case of a model where the risky asset is driven by a spectrally negative Lévy process. For this class of Lévy processes, we review theory on first exit times of finite and semi-infinite intervals. Subsequently, we determine the Laplace transform of the exit time and exit position from an interval containing the origin of the process reflected at its supremum. The proof relies on the application of Itô-excursion theory to the excursions of the reflected process away from zero. Combining the obtained results with martingale methods, we solve for the optimal stopping problem connected to the valuation of American perpetual put and Russian option and their Canadized versions, where we simply assumed the equivalent martingale measure already to have been chosen for us.

The fourth chapter complements the study on Lévy processes without negative jumps of the previous chapter. We find the Laplace transform of the first exit time of a finite interval containing the origin of the process reflected at its infimum. Then we turn our attention to these reflected processes killed upon leaving a finite interval containing zero and determine their resolvent measures. Invoking the R-theory of irreducible Markov chains developed by Tuomen and Tweedie [124], we are able to give a relatively complete description of the ergodic behaviour of their transition probabilities. The obtained results on Lévy processes in this and the previous chapter have also applications in the context of the theories of queueing, dams and insurance risk.

Finally, the fifth chapter considers the utility-optimisation problem of an agent that operates in a general semimartingale market and seeks to trade so as to maximise his utility from inter-temporal consumption and final wealth. In this setting existence is established following a direct variational approach, invoking a famous result of Komlós [80]. Also a characterisation for the optimal consumption and final wealth plan is given. The earlier mentioned problem of pricing contingent claims can be treated in this framework.

Publication details

The first four chapters presented in this thesis have been submitted to or accepted by refereed journals. The first chapter is a joint work with Andreas Kyprianou and has been accepted for publication in *the Annals of Applied Probability* as

> Kyprianou, A.E. and Pistorius, M.R. Perpetual options and Canadization through fluctuation theory.

Chapter 2 came out of a project with Søren Asmussen and Florin Avram and has been submitted to *Stochastic Processes and their Applications*.

> Asmussen, S., Avram, F. and Pistorius, M.R. American and Russian options under exponential phase type Lévy models.

The third chapter was written jointly with Florin Avram and Andreas Kyprianou and has been accepted in abbreviated form for publication in *the Annals of Applied Probability* as

> Avram, F., Kyprianou, A.E. and Pistorius, M.R. Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options.

The fourth and fifth chapter have been submitted to *Journal of Theoretical Probability* and *Journal of Economic Theory* respectively.

> Pistorius, M.R. On exit and ergodicity of the reflected spectrally negative Lévy process reflected at its infimum.

> Pistorius, M.R. A direct approach to existence and characterisation of optimal consumption and investment in semimartingale markets.