Chapter 8

The Summary, the Theses and Conclusions

8.1 Introduction

In our last chapter three items will come up:

– *First*, Brouwer's own summary of his dissertation on page 179 and 180. It only makes one important assertion: *Mathematics is a free creation of the individual mind.* Since the expression 'free creation' is used by more mathematicians, it is interesting and important to make a comparison, and see in what respect Brouwer was different. The term 'free creation' as an act also reminds us of the 'two acts of intuitionism'.

– *Second*, the theses, added at the end of the dissertation. There are several known drafts for this list. Of special interest in regard to this is a letter to De Vries, in which some of the theses are clarified. Especially the thesis about the existence of the actual infinite (one that appeared only in one of the drafts for the list) and its accompanying elucidation, is of major importance. Seemingly conflicting statements about the infinite can be read in the dissertation, as well as in the notebooks. An interpretation has to be searched for, and can be found in connection with the interpretation for a set or a sequence to be 'actually finished' or 'potentially finished'.

– *Third*, a summary of our conclusions from the different chapters.

8.2 The summary of Brouwer's dissertation

After having finished the discussion of the four examples that substantiated his view on the role of logic in mathematics, the last two pages of Brouwer's dissertation are devoted to some concluding remarks about the three chapters that compose his dissertation.

On page 179 and 180 the central ideas of the three chapters are presented in the form of a very compact summary about what mathematics is, and what it is not. It opens as follows:

> Mathematics is a free creation, independent of experience; it develops from a single aprioristic ur-intuition, which may be called *invariance in change* as well as *unity in multitude*.[1]

But more mathematicians have characterized mathematics as a 'free creation', and therefore we compare Brouwer's opinion with the views of two others. The difference between Brouwer on the one hand, and Cantor and Dedekind on the other, is striking.

Cantor's 'freie Entwickelung'

In § 8 of the *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* from 1883, Cantor dealt with two interpretations of the concept of 'existence of integers',[2] and according to Cantor these two interpretations always occur together, which is caused by the fact that they both find their origin

> in der *Einheit* des *Alls, zu welchem wir selbst mitgehören.* – Der Hinweis auf diesen Zusammenhang hat nun hier den Zweck, eine mir sehr wichtig scheinende Konsequentz für die Mathematik daraus herzuleiten, daß nämlich letztere bei der Ausbildung ihres Ideenmaterials *einzig* und *allein* auf die *immanente* Realität ihrer Begriffe Rücksicht zu nehmen und daher *keinerlei* Verbindlichkeit hat, sie auch nach ihrer *transienten* Realität zu prüfen.[3]

which distinguishes mathematics from all other sciences; a consequence of this is the following:

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[1] dissertation, page 179: De wiskunde is een vrije schepping, onafhankelijk van de ervaring; zij ontwikkelt zich uit een enkele aprioristische oer-intuïtie, die men zowel kan noemen *constantheid in wisseling* als *eenheid in veelheid.*

[2] These two interpretations are:

> 1. 'Einmal dürfen wir die ganzen Zahlen insofern für wirklich ansehen, als sie auf Grund von Definitionen in unserm Verstande einen ganz bestimmten Platz einnehmen', von allen übrigen Bestandteilen unseres Denkens aufs beste unterschieden werden, zu ihnen in bestimmten Beziehungen stehen und somit die Substanz uneres Geistes in bestimmter Weise modifizieren.

Cantor called this form of reality of the numbers the *intrasubjective* or *immanent* reality. The second mode of ascribing reality to numbers is:

> 2. 'als sie für einen Ausdruck oder ein Abbild von Vorgängen und Beziehungen in der dem Intellekt gegenüberstehenden Aussenwelt gehalten werden müssen.

Cantor called this the *transsubjective* or *transient* reality.

[3] [Cantor 1883], page 19.

> Die Mathematik ist in ihrer Entwickelung völlig frei und nur an die selbstredende Rücksicht gebunden, daß ihre Begriffe sowohl in sich widerspruchslos sind, als auch in festen durch Definitionen geordneten Beziehungen zu den vorher gebildeten, bereits vorhandenen und bewährten Begriffen stehen. Im besondern ist sie bei der Einführung neuer Zahlen nur verpflichtet, Definitionen von ihnen zu geben, durch welche ihnen eine solche Bestimmtheit und unter Umständen eine solche Beziehung zu den älteren Zahlen verliehen wird, daß sie sich in gegebenen Fällen unter einander bestimmt unterscheiden lassen. Sobald eine Zahl allen diesen Bedingungen genügt, kann und muß sie als existent und real in der Mathematik betrachtet werden.[4]

According to Cantor, this freedom is in no way a threat to science, thanks to the small margin for arbitrariness, together with its selfcorrecting structure. Any further limitation by rules is not required:

> denn das *Wesen* der *Mathematik* liegt gerade in ihrer *Freiheit*.

Hence also for Cantor the system of the natural numbers is the result of a 'freie Entwickelung', but, different from Brouwer, this free development is not purely based on the ur-intuition alone, and is not the result of a construction. Cantor's freedom is a total freedom within the constraints of consistency of its concepts, internally *and* in their mutual relations. For Brouwer, freedom is expressed by the admissibility of *any* mathematical construction which is solely based on the ur-intuition.

Dedekind and 'freie Schöpfung'

Also Dedekind, in the preface to the first edition of *Was sind und was sollen die Zahlen*, claimed that the system of the natural numbers is the result of a free creation of the human mind:

> Indem ich die Arithmetik (Algebra, Analysis) nur einen Teil der Logik nenne, spreche ich schon aus, daß ich den Zahlbegriff für gänzlich unabhängig von den Vorstellungen oder Anschauungen des Raumes und der Zeit, daß ich ihn vielmehr für einen unmittelbaren Ausfluß der reinen Denkgesetze halte. Meine Hauptantwort auf die im Titel dieser Schrift gestellte Frage lautet: die Zahlen sind freie Schöpfungen des menschlichen Geistes, sie dienen als ein Mittel, um die Verschiedenheit der Dinge leichter und schärfer aufzufassen.[5]

In this quote Dedekind appears as a logicist: Arithmetic is just a part of logic. So neither with Dedekind the system of the natural numbers is the result of a construction, based on the ur-intuition of mathematics; for Dedekind the

[4] loc. cit. page 19.

[5] [Dedekind 1930b], page V. This paragraph was quoted earlier; see chapter 1, page 22.

system ℕ is based on the concepts of 'mapping' and 'chain'.[6] The existence of infinite systems is proved by means of the theorem 'es gibt unendliche Systeme', in the proof of which Brouwer noticed a looming paradox, which he mentioned in the third notebook, and which we quoted earlier on page 293.

Brouwer's free creation

Brouwer brought back the basis and foundation of all mathematics to its most primitive form, the ur-intuition of 'invariance in change' or 'unity in multitude'.[7] The interpretation of Brouwer's *free creation* should be the following: in this creation the ur-intuition is the most fundamental element, and, departing from this the construction of the mathematical building is a free creative act, as this was sketched in the previous chapters of this dissertation.

See for instance our page 44, the *construction of the natural numbers*, in which the abstraction from the content of an event or a sequence of events is an *act*, thus making a number independent of the nature and content of such a sequence.

Another example is the construction of the rational numbers, in which the time span (the continuous interval) between any two consecutive events is identical in character to the time interval between any other two successive events. This is neither the result of a discovery, nor a corollary of an axiom, but it is the result of an *act* of our mind. We *force* them as it were to be similar (see page 46).

And again we perceive this free creation in the construction of the *measurable continuum* (page 79, item 7), which is, after ω times the splitting of every interval into two parts, *made to be everywhere dense by our own act* of contracting every unpenetrated segment into one point:

> But we agree to contract every segment not penetrated by the scale into one point, in other words, we consider two points as different only when their approximating dual fractions differ after a finite number of digits.[8]

A fourth example of mathematics as a free creation is thesis II, in which the principle of complete induction is declared to be neither a theorem, nor an axiom, but simply an *act* in the mathematical construction (see the next section).

[6]Compare also page 19. Clearly, for Dedekind the number concept is not depending on space or time, as this is also the case with Cantor; see page 8.

[7]In a footnote to the quoted beginning of the summary, Brouwer mentioned F. Meyer, who stated in the *Verhandlungen des Heidelberger Kongresses* that one thing will be sufficient, since the act of thinking that thing automatically includes a second thing, viz. the act of thinking itself. This is contested by Brouwer, since Meyer presupposed in his argument the intuition of *two*.

[8]dissertation, page 10: Maar we spreken af, dat we elk segment, waarin de schaal niet doordringt, tot een enkel punt denken samengetrokken, m.a.w. we stellen twee punten alleen dán verschillend, als hun duale benaderingsbreuken na een eindig aantal cijfers gaan verschillen.

These examples show best Brouwer's view of mathematics as the result of the free creative act.[9] In the continuation of the summary of his dissertation, Brouwer claimed that also the projection of mathematical systems on the experience of our environment is a free act. In the discussion of the second chapter of Brouwer's dissertation (chapter 6 of this dissertation) we dealt with man's faculty of taking a mathematical view of his life. Man observes regularity in the world, he observes recurring sequences of events and he discovers the possibility of expressing these sequences in a mathematical way. He also discovers methods of early interference in these sequences, thus changing its course into a desired direction. The *free act* consists of the creation of a mathematical model of the physical world, we *force* nature into such a model, in order to rule the surrounding world for our own well-being.

> In this respect one mathematical system can appear more practical, more economical than another, at least relative to a definite kind of purpose which one wishes to attain: none of them is absolutely efficient.[10]

The last two paragraphs of Brouwer's dissertation once more emphasize the difference between mathematics, its accompanying language and logic.

The *second last* clearly states Brouwer's (by now well-known) dictum about the employed language when expressing mathematical statements:

> In mathematics, mathematical definitions and properties ought not to be studied again by mathematical methods; they ought to be no more than a means of conducting as economically as possible one's own memory and communication with other people.[11]

In the definitions (and generally in all mathematical language) there are primitive and irreducible concepts like *continuous, entity, once more, and so on.* These concepts are elements of construction, immediately perceived in the ur-intuition of the continuum. This paragraph clearly rejects any form of meta-mathematics, i.e. mathematics about the structures of mathematical language instead of mathematics itself. This form of meta-mathematics has as objects mathematical *words*, it has as relations the *rules*, according to which these words

[9]Compare this with a quote from E.W. Beth's *Modern Logic*, who, when discussing the status of mathematical knowledge in the several philosophical movements, noted about intuitionism:

> The intuitionist conceives it as a form of self-knowledge. ([Beth 1967], page 102: De intuïtionist vat haar op als een vorm van zelfkennis.)

[10]Dissertation, page 180: Het ene wiskundige systeem kan daarbij praktischer, economischer blijken, dan het andere, althans voorzover betreft een bepaalde categorie van doeleinden, die men door middel van die systemen tracht te bereiken: absoluut doeltreffend zijn ze geen van alle.

[11]dissertation, page 180: In de wiskunde behoren wiskundige definities en eigenschappen niet zelf weer wiskundig te worden bekeken, maar alleen een middel te zijn, om eigen herinnering of mededeling aan anderen van een wiskundig gebouw zo economisch mogelijk te leiden.

can be grouped into meaningful sentences, and it has as results *statements about the language of mathematics*, about logistics, but not about mathematics itself.

The *last* paragraph deals with the impossibility of a construction of the mathematical building on the foundation of logic alone, without any mathematical intuition. Again, one is doing theoretical logic, or, at the best, logistics, but certainly not mathematics. One is just constructing a language-building:

> A logical construction of mathematics, independent of the mathematical intuition, is impossible – for by this method no more is obtained than a linguistic structure, which irrevocably remains separated from mathematics – and moreover it is a contradictio in terminis – because a logical system needs the basic intuition of mathematics as much as mathematics itself needs it.[12]

As a final conclusion we can say that we know that metamathematics and mathematical logic are not themselves methods of constructing mathematics, but merely the observation and the study of the accompanying language of a mathematical construction.

8.3 The theses

Until recent time a compulsory list of theses formed an integral part of the dissertation and had to be defended together with it. Brouwer's theses, 21 in number, all have a philosophical and/or mathematical content and consist for a great part of the conclusions from the several topics that were discussed in the dissertation. Here are some examples:

> (II) It is not only impossible to prove the admissibility of complete induction, but it ought neither to be considered as a special axiom nor as a special intuitive truth. Complete induction is an act of mathematical construction, already justified by the basic intuition of mathematics.[13]
>
> *[referring to conclusions from chapter I:]*

[12] dissertation, page 180, the last paragraph: Een logische opbouw der wiskunde, onafhankelijk van de wiskundige intuïtie, is onmogelijk – daar op die manier slechts een taalgebouw wordt verkregen, dat van de eigenlijke wiskunde onherroepelijk gescheiden blijft – en bovendien een contradictio in terminis – daar een logisch systeem, zo goed als de wiskunde zelf, de wiskundige oer-intuïtie nodig heeft.

[13] De geoorloofdheid der volledige inductie kan niet alleen niet worden bewezen, maar behoort ook geen plaats als afzonderlijk axioma of afzonderlijk ingeziene intuïve waarheid in te nemen. Volledige inductie is een daad van wiskundig bouwen, die in de oer-intuïtie der wiskunde reeds haar rechtvaardiging heeft.

(Compare this to Poincaré in [Poincaré 1916], chapter I, section V: consistency of complete induction cannot be proved; it is a 'propriété de l'esprit lui-même'.)

> (V) The arithmetical operations on the measurable continuum ought to be defined by means of group theory.[14]
>
> *[referring to conclusions from chapter II:]*
>
> (VII) Attributing 'objectivity' to physical notions like *mass* and *number* is based upon their invariability with respect to an important group of phenomena in the mathematical image of nature.[15]
>
> (VIII) Human understanding is based upon the construction of common mathematical systems, in such a way that for each individual an element of life is connected with the same element of such a system.[16]
>
> *[referring to conclusions from chapter III:]*
>
> (IX) Mathematics is independent of logic; practical logic and theoretical logic are applications of different parts of mathematics.[17]
>
> (XII) Besides the finite there are no other cardinalities than: denumerably infinite, denumerably infinite unfinished, continuous.[18]
>
> (XIII) Cantor's second number class does not exist.[19]

Several other theses are about potential theory, a subject which was not discussed in the dissertation but to which Brouwer devoted several of his earliest papers.

An interesting aspect is formed by the different drafts for the theses to which often clarifying notes were added; many of the draft-theses did not find their way into the dissertation.[20] Some interesting (and sometimes puzzling) observations can be made in the different drafts, especially in combination with a draft-letter to J. de Vries, in which Brouwer elucidated the main aspects of his dissertation.[21] This elucidation is done in four sections, and each section ends with a reference to relevant theses from the list of 21, followed by the page numbers from the dissertation to which these theses refer; however, often one or more 'theses in plain language' (that is, not specifically denominated as a thesis)

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[14] De hoofdbewerkingen op het meetbaar continuüm behoren door groepentheorie te worden gedefinieerd.

[15] Het toekennen van 'objectiviteit' aan fysische grootheden als *massa* en *aantal* berust op de invariabiliteit daarvan bij een belangrijke groep van verschijnselen in het wiskundig natuurbeeld.

[16] De verstandhouding der mensen berust op het bouwen van gemeenschappelijke wiskundige systemen, en het verbinden aan eenzelfde element van zulk een systeem van een levenselement voor elk der individuen.

[17] Wiskunde is onafhankelijk van logica; practische logica en theoretische logica zijn toepassingen van verschillende gedeelten der wiskunde.

[18] Behalve de eindige, bestaan er geen andere machtigheden dan: aftelbaar oneindig, aftelbaar oneindig onaf, continu.

[19] De tweede getalklasse van Cantor bestaat niet.

[20] These theses and their clarifying notes can be found in the new edition of Brouwer's dissertation, [Dalen 2001].

[21] Professor J. de Vries (1858-1938), Utrecht University. The letter is undated, but most likely it is from shortly after his public defence in February 1907.

are added, and these latter are always specimen from the several drafts which did not end up in the dissertation. They apparently were removed from the final list, possibly under Korteweg's influence, either because of their content or simply because the list became too long and a choice had to be made. But they were certainly *not* removed because of a change in Brouwer's opinion, since they were explicitly mentioned in the letter to J. de Vries, including relevant page references.
The four different sections in the letter to De Vries are:

A. *About the classification of mathematics as a special branch of logic.* It is impossible to classify mathematics under logic, since in case of an attempted proof of a mathematical truth from logic, that mathematical truth is tacitly and intuitively presupposed in the deduction. One of the theses added to this conclusion is the unpublished one from the second draft:

> In a logical treatment of mathematics there is nothing against the petitio principii, provided it is read from the intuition. (see page 176) [*of Brouwer's dissertation*][22]

This item was discussed on page 305, in Poincaré's criticism towards Hilbert.

B. *About the actual execution of the intuitive construction.* This was mainly treated in the first chapter, and partly in chapter 3. Among the relevant theses there is an unpublished one, from draft 2:

> A strict separation should be made between the intuitive time and the scientific time.[23]

This thesis is added as a footnote to Brouwer's main conclusion on mathematical intuition (*'The only a priori element in science is time'*) on page 99 of his dissertation.

C. *About the general character of science and the relation between mathematics and other sciences.* This mainly refers to the second chapter of Brouwer's dissertation. Science consists of the projection of mathematics on our world of experience, which seems peculiar, since mathematics is not depending on any daily experience; we force a mathematical description on nature instead. The added thesis, which is not from the dissertation-list, is thesis 26 (not verbatim) from an extra list with additional clarifications:

> Mathematics is not a science like other sciences, but it is a moral act consisting of doing science.[24]

A similar thesis is XXIV from draft 2:

[22] Bij een logische behandeling der wiskunde is niets tegen een petitio principii, mits die uit de intuïtie wordt afgelezen.

[23] Men behoort streng te onderscheiden tussen de intuïtieve en de wetenschappelijke tijd.

[24] Wiskunde is niet een wetenschap als een andere, maar een morele daad die het bedrijven van de wetenschap is.

> Mathematics should not be considered as a science as any other, but as a medium to the different sciences.[25]

D. *About the question whether or not actual infinite sets exist.*[26] This question was also dealt with on our page 78, as well as in the discussion of Poincaré's critique on Hilbert (see page 306); see also the quote from Aristotle in the beginning of chapter 3. Poincaré, on the one hand, completely rejected the actual infinite; Cantor, on the other hand, admitted infinities of always higher cardinalities. In his letter to De Vries, the *actual* infinite is restricted by Brouwer to the following sets:

> I acknowledge denumerably infinite sets, and with a restriction, the continuous cardinality, and finally, with another restriction, a new cardinality, which I call denumerably infinite unfinished. I expose however, all the higher cardinalities of Cantor as a logical chimera. At the same time I try to strip transfinite set theory of its parasite parts, such as transfinite exponentiation, the theorem of Bernstein with its applications, and more; all of which result from the false logical foundations of set theory. In this connection I can formulate:
>
> 1. Actual infinite sets can be created mathematically, even though in the practical applications of mathematics in the world only finite sets occur.[27]

This last claim is also the first thesis from the first draft of unpublished theses, and we notice that, on this point, Brouwer seems not always to be clear, consistent and unambiguous in his texts. Brouwer knew of course that actual infinities exist, for instance the system of the natural numbers or that of the rational numbers; the problem for him was to subsume them in a mathematical construction.

In the dissertation (page 176, in the discussion of Poincaré's comment) also the 'actual infinite of the Cantorians' is said to exist, but here it is explicitly restricted to that 'which can be intuitively constructed'. This can only refer to the denumerably infinite and possibly also the denumerably infinite unfinished cardinality (under the proper interpretation of the latter). The continuum is

[25] Wiskunde behoort niet te worden beschouwd als een wetenschap als een andere, maar als het medium tot de verschillende wetenschappen.

[26] Note that in 1885 Cantor published in the *Zeitschrift für philosophie und philosophische Kritik* a paper, *Über die verschiedene Standpunkte in bezug auf das aktuelle Unendliche*, which is of historical interest. See [Cantor 1932], page 371.

[27] (English translation by D. van Dalen in [Dalen, D. van 1999], page 118. [Ik] erken aftelbaar oneindige verzamelingen, en met een restrictie de continue machtigheid, en ten slotte met een andere restrictie een nieuwe machtigheid, die ik noem aftelbaar oneindig onaf. Alle hogere machtigheden van Cantor echter toon ik aan als logische hersenschimmen. Tegelijk tracht ik de transfiniete Mengenlehre van haar parasitaire gedeelten als transfiniete machtsverheffing, theorema van Bernstein met zijn toepassingen, en meer, die alle uit de valse logische grondslagen ervan voorkomen, te ontdoen. Ik kan in dit verband formuleren: 1. 'Actueel oneindige verzamelingen zijn wiskundig te scheppen, ook al treden bij de practische toepassing der wiskunde in de wereld slechts eindige verzamelingen op' (Zie page 120, 142-143).

not intuitively constructed since this is given to us in its entirety. Nevertheless it is subsumed under one of Brouwer's four possible cardinalities and it certainly cannot be regarded as finite. Also in the letter to De Vries the continuum is implicitly included as one of the 'actual infinite' sets. But from the dissertation it follows that the continuum is no point set, it is only the matrix, *onto* which denumerably infinite many points can be constructed; it is an *infinite source*. Apparently, this goes very well together for Brouwer.

In spite of Brouwer's claim quoted above that 'in the practical applications in the world only finite sets occur, one should of course suppose an *actual* infinite set to be completed, but on page 9 and 10 of his dissertation, Brouwer pointed out that a denumerable set, which is by definition given by some algorithm, may *not* be considered as an example of a finished totality. On these pages, where the construction of a scale on the intuitive continuum is discussed, we also find the method of approximation of some arbitrary point. This method is of relevance to the concept of the actual infinite, since, when selecting a point P, we can approximate this point, without ever reaching it, by an infinite dual fraction (which can be viewed as an infinite sequence of dual fractions),

> (...) given by an arbitrary given law of progression, (...) However, we can never consider the approximating sequence of a *given definite* point as being *completed*, so we must consider it as partly unknown.[28]

In a handwritten correction to his own copy of the dissertation, Brouwer even added as an example: 'take for instance the number π'.[29] Thus a *lawlike* sequence of progression for the number π has to be regarded as partially unknown, whereas every element of this sequence can be computed directly and unambiguously.

Also the notebooks contain (more or less) conflicting remarks on the actual infinite; e.g. on the one hand:

> (VIII–20) I can think a fundamental sequence as *finished*, just as (the value of) a convergent sequence (the first one gives certainty of the equality of the terms, the second of the limiting value).[30]

So, considering Brouwer's concept of a fundamental sequence, viz. any sequence of ordertype ω,[31] together with the fact that every well-defined, i.e. algorithmically given, denumerable set can be given in the form of a fundamental sequence, results in such a set as an example of an 'actual infinity'.

But, on the other hand, in the same notebook we find the following paragraph:

[28] [gegeven] volgens een willekeurige denkbare voortschrijdingswet (...) We kunnen de benaderingsreeks van een *bepaald aangewezen* punt evenwel nooit *af* denken, dus moeten haar als gedeeltelijk onbekend beschouwen.

[29] See also the discussion on page 77.

[30] Een fundamentaalreeks kan ik 'af'denken; eveneens de waarde van een convergente reeks (de eerste geeft de vastigheid van de gelijkheid der termen, de tweede die van de limiet waarde).

[31] See for instance the *Berliner Gastvorlesung*, [Dalen, D. van (ed.) 1992], page 31.

> (VIII–24) One should always keep in mind that ω only makes sense as a living and growing induction in motion; as a stationary abstract entity it is senseless; ω may never be conceived to be finished, as a new entity to operate on; however you may conceive it to be finished in the sense of turning away from it while it continues growing, and to think of something new.[32]

This, again, is clear, but it seems to be in conflict with the preceding quote: even the set of the natural numbers should in this option be considered as a 'for ever unfinished and always growing' sequence.

The conclusion from the letter to J. de Vries is unambiguous: the actual infinite does exist and the conclusion from page 176 of the dissertation is the same; moreover there are other places in his dissertation where Brouwer presented direct or indirect arguments for the existence of the actual infinite. Whereas on page 9 the phrase 'it is easy to construct on the continuum a sequence of points having the order type of the positive and negative whole numbers' still can be interpreted as expressing a process of never terminating growth, the sentence on page 62:

> The mathematical intuition is unable to create other than denumerable sets of individuals. But it is able, *after having created* a scale of order type η (...)[33]

The expression '*after* having created' *seems* to refer to a finished, actually denumerable set. On Brouwer's page 142 it is expressed as follows (we quoted and discussed this earlier in a different context; see our page 78):

> In the first chapter we have seen that there exist no other sets than finite and denumerably infinite sets and continua; this has been shown on the basis of the intuitively clear fact that in mathematics we can create only finite sequences, further by means of the clearly conceived 'and so on' the order type ω, but only consisting of equal elements; (consequently we can, for instance, never imagine *arbitrary* infinite dual fractions as finished, nor as individualized, since the denumerably infinite sequence of digits cannot be considered as a denumerable sequence of *equal* objects), and finally the intuitive continuum, (...)[34]

[32] Men bedenke steeds dat ω alleen zin heeft, zolang het leeft, als groeiende, bewegende inductie; als stilstaand abstract iets is het zinloos; zo mag ω nooit àf gedacht worden, om m.b.v. het *geheel* als nieuwe eenheid te werken: wel mag je het àf denken in de zin, van je er van af te keren, terwijl het doorloopt, en iets nieuws te gaan denken.

[33] De mathematische intuïtie is niet in staat andes dan aftelbare hoeveelheden geïndividualiseerd te scheppen. Maar wel kan zij, eenmaal een schaal van het orde type η opgebouwd hebbend, (...)[*my emphasis in the main text*].

[34] We hebben in het eerste hoofdstuk gezien, dat er geen andere verzamelingen bestaan, dan eindige en aftelbaar oneindige, en continua; hetgeen is aangetoond op grond van de intuïtieve waarheid, dat wij wiskundig niet anders kunnen scheppen, dan eindige rijen, verder op grond van het duidelijk gedachte 'en zovoort' het orde type ω, doch alleen bestaande uit

Hence *only* arbitrary infinite dual fractions (a choice sequence in dual representation) cannot be imagined as finished and *thus*, one should say, a lawlike sequence may be regarded as finished, giving an actual infinity.

Brouwer's view is unambiguously expressed in the letter to De Vries: an actually infinite set exists if this notion is limited to algorithmically constructed denumerable sets. Also the continuum exists as an intuitively given actuality, but no sets of higher cardinality than denumerable can be constructed and therefore do not exist. Hence this also applies to other 'Cantorian' sets like the set of all subsets of a denumerably infinite set since it cannot be defined by an algorithm.

But then, how should we construe the conflicting quotes from page 9 and 10 of the dissertation and from the paragraph from notebook VIII, page 20? How is it possible that, on the one hand, the lawlike sequence of π must be considered to be partially unknown, and, on the other hand, (VIII–20) 'I can think a fundamental sequence as finished'? When is an infinite set finished and when does the actual infinite exist; and when are the terms of an infinite expansion known or unknown. Of course, arbitrary infinite dual fractions' (hence non-lawlike choice sequences) are only known as far as the choices are actually made and therefore they are in their totality unknown on principle.

Different interpretations seem to be possible for 'finished' and 'unfinished', as well as for 'known' and 'unknown'. In an attempt to create order in these seemingly conflicting remarks by Brouwer, we propose the following interpretations for these concepts. They do, we are convinced, justice to Brouwer's views; we are even inclined to imagine that he would have offered the same mathematical exegesis.

In regard to the concept 'finished': When constructing for instance the system ω (or η) in a systematic algorithmic manner, then the result of the actual construction of the elements, one by one, forms of course a never terminating and always finite sequence. The process is never actually finished. But, because of the repeated application of the same algorithm, because of the always equal steps, we may *declare* the set ω to be finished. This is another example of mathematics as an act, as a free creation of the mind: we may jump as it were over the whole procedure of the successor operation. Just as we may consider a first and a second event and their connecting continuum together as one single event, retained in memory as such and separated by a time span from a new event, thus constructing the system $\mathbb{N}$, we may consider ω as one single unit, as one 'experienced event', and add a new element, called $\omega + 1$. We *consider* or *idealize* the actual infinite set to exist and we have arguments for this *act* since we can, without hesitation, mention every member of the set (but of course not *all* members!), exactly because of the simplicity and the constancy of the algorithm.

gelijke elementen, zodat we ons b.v. de *willekeurige* oneindige duaalbreuken nooit af, dus nooit geïndividualiseerd kunnen denken, omdat het aftelbaar oneindig aantal cijfers achter de komma niet is te zien als een aftelbaar aantal *gelijke* dingen, en tenslotte het intuïtief continuüm, (...)

We can put this in the following terms: *extensionally* speaking the actual infinite never exists, since an extensionally given set is supposed to be the result of a proper mathematical construction of its individual terms, including its termination, which is impossible. But we may also conceive the actual infinite to be an *intensional* mathematical object, e.g. the system of the natural numbers defined by the successor operation. This makes the quotes given above from VIII–24 comprehensible: as a stationary abstract entity it is senseless, but you may turn away from it and let it grow while doing something else. That 'doing something else' may then, for instance, consist of the continuation of the act of counting from $\omega + 1$ onwards. In that sense you may conceive it to be finished. The intensional definition of the algorithm makes the free act of declaring it to be finished defensible, and this makes the claim that the actual infinite exists equally defensible. Intensionally it is there; see e.g. also the following quote from the seventh notebook:

> (VII–16) ω is finished by our innate mathematical induction.[35]

Now the only remaining quote that does not fit in this picture, is the one from page 10 of the dissertation, so it seems:

> However, we can never consider the approximating sequence of a *given definite* point as being *completed*, so we must consider it as partly unknown.

and this is directly connected with the question of 'when is something known or not known'. Clearly, judging by the quotes from the dissertation and from the notebooks, several interpretations are possible. Of course the unchosen terms of a non-lawlike choice sequence are unknown on principle; they remain so as long as no choice has been made. But a different form of 'unknown' must be meant by Brouwer in the quote given above from the dissertation. Even for a lawlike sequence like the one for π, the uncomputed terms are unknown (for the time being), even if they are known in principle, even if a computer can fix its value in an instant; as long as the computation is not actually performed we do not yet know its outcome; this has to be understood in its most basic and primitive sense. Clearly this form of 'unknown' is closely linked with the extensional definition for the sequence of π, since these two concepts are used in one and the same sentence.

It may be confusing to the reader that Brouwer employed two interpretations for the 'actual infinite' and two interpretations for 'unknown', in a seemingly random way, but close reading of the several quotes reveals which interpretation Brouwer exactly had in mind on that specific occasion.

When the actual infinite is declared to exist (which we interpreted to be the result of a free creating act of the mind), i.e. in case of an intensional definition, the elements of the set or the terms of the sequence may be declared to be known. However, if an infinite quantity is extensionally defined and therefore never finished, then only the finite finished part of the elements is known.

[35] ω is af door de mathematische inductie, die in ons is.

8.4 Summary of our conclusions

Many interpretations or re-interpretations were made in the course of the discussion of the several topics from Brouwer's dissertation. Many conclusions were drawn at the end of each of the previous chapters or their sections or subsections. In the following we will present a summary of the most relevant items in regard to the foundational aspects of mathematics.

– *The ur-intuition and the construction of the ω-scale* (page 44).

Two well-separated and actually experienced events, combined into the unity *event – connectiong medium (time span) – next event*, and divested of all quality, form the beginning of the ω-scale (i.e. the numbers 'zero' and 'one') to which unity can be added a new event, well-separated from that unity and which forms, after abstraction, the number 'two', etc. Hence, what is retained in memory is a *sign*, which stands for the result of that abstraction.

– *The status of this sign* (page 56).

This representing sign is also a mental construction in the form of an abstract symbol, and does not belong to the accompanying language yet. Only its oral or written expression belongs to that language.

– *The construction of the η-scale* (page 46).

The experience of the flowing, of the connecting medium between a first event ('zero') and a next ('one') can itself be seen as an event and therewith it *is* the first intercalated element of the η-scale (the element 'half'), and, since this 'flowing' is experienced as well-separated form the events 'zero' and 'one', it is in its turn again connected by a flowing with both, zero and one. Hence the procedure of intercalation can be repeated indefinitely.

– *The scale of integers* (page 48).

If we call the second event 'zero' and the third 'one' (we are free to do so), then we may call the intercalation between the first and the second event 'minus one'; the second intercalation between the first event and 'minus one' we call 'minus two', etc. The first event then (informally) becomes 'minus infinity'.

– *The everywhere dense η-scale* (page 79).

This scale on the 'intuitive continuum in a graphic (i.e. geometrical) representation' is the result of the free act of contracting every not-penetrated segment of this continuum into one point; that is, identifying two points which do not differ after any finite number of decimal places, as one and the same point.

– *The Bolzano-Weierstrass theorem* (page 81).

In Brouwer's argumentation for this theorem, the principle of the excluded middle is employed. Attempts to prove it in a strictly constructive way turn out to be unsuccessful.

– *Covering by, or completion to a continuum of an everywhere dense scale* (page 120).

This can take place by mapping the set of the rationals on a continuum on which an everywhere dense scale is constructed.

– *The third construction rule for sets* (page 122).

Brouwer's choice for this rule as one of the possibilities to construct a set, must be the result of the great influence that Cantor still had on the young Brouwer, and was, with good reasons, later on rejected by him in the *Addenda and Corrigenda* (page 130).

– *Brouwer's solution to the continuum problem* (chapter 5).

His solution was the only possible and almost trivial one in view of his constructive approach of sets of points on the continuum, but it did not answer Cantor's conjecture that $2^{\aleph_0} = \aleph_1$, which is also impossible because Brouwer did not recognize any aleph's, apart from $\aleph_0$.

– *Brouwer's view on physics* (page 181).

His stern view on the moral aspects of physical practice was the result of his pessimistic outlook on mankind. Man's only desire is, according to Brouwer, to rule and to increase his power. The result is an approach towards physics, which is certainly not common among physicists.

– *Objectivity and apriority* (page 210)

Brouwer's concept of objectivity turned out to be a direct and natural corollary of his solipsism. Also his view on apriority is a result of the ur-intuition, which makes, in contrast with Kant, space superfluous as an apriorisitc element in the construction of mathematics.

– *The role of logic* (page 228)

This role is reduced to that of a set of rules for the accompanying linguistic reasoning in the construction of the mathematical building. The principium tertii exclusi is ultimately rejected for infinite sets.

– *The hypothetical judgement in mathematics* (page 230).

This judgement is subject to stricter rules than the ones from the later BHK proof interpretation. The premise of the judgement has to be the result of a properly performed mathematical construction. This was clarified with the help of several examples.

– *The denumerably infinite unfinished cardinality* (page 266).

This is the third in the list of possible cardinalities for sets. This cardinality and in particular 'Brouwer's lemma' can only be properly interpreted and understood under a stricter and more limiting set of conditions than the ones given in Brouwer's dissertation.

– *Brouwer and Gödel's first incompleteness theorem* (page 286).

A single remark, made by Brouwer in one of the notebooks:

> (VIII–44) The totality of mathematical theorems also, among other things, constitutes a set, which is denumerable but never finished.

can be interpreted and defended as a foreshadowing of Gödel's first incompleteness theorem.

– *The actual infinite* (page 320).

The concepts 'finished/unfinished' and 'known/unknown' as well as the existence of the 'actual infinite' were discussed in this chapter, with the conclusion that, for a proper interpretation, a distinction has to be made between an extensional and an intensional definition of a set.

We also mentioned the *two acts of intuitionism* (page 57), explicitly expressed in Brouwer's later work, viz.
1) the strict separation between mathematics and its accompanying language. Mathematics is fundamentally languageless,
and
2) the actual construction of mathematics, strictly separated from any language, but solely based on the ur-intuition. In this construction a set becomes a *law*.

We clearly recognized both of Brouwer's acts already in his dissertation, without, however, denoting them as such. See for this for instance our page 57 and chapter 4, which stipulate the construction of the elements for a set.

The *spread concept* could be identified in chapter 1 of Brouwer's dissertation (page 117 of this dissertation), in which its role is still limited to decide whether or not a set is dense in some specific interval, instead of using it for the construction of set elements in the form of choice sequences.

Choice sequences did not yet appear in the dissertation, but we met them several times in the notebooks, often in the form of thought experiments. We can observe similar experiments with other mathematicians (see for instance [Borel 1908b], page 16, or [Borel 1950], page 160), but, whereas with others it did not lead to anything revolutionary new notions, the choice sequence concept in the notebooks clearly were a foreboding of developments ahead.

This and similar forebodings remain one of the fascinating aspects of reading in the dissertation in combination with the notebooks. Intuitionistic mathematics had not yet matured in those early days, but the signs were there; all was waiting for a breakthrough, which was to come in 1917, with far-reaching consequences.

Probably a further and more detailed study of the nine notebooks, in combination with Brouwer's later work, will reveal more seeds of later developments. An annotated publication of the notebooks is in preparation.