Preface

Almost a century after L.E.J. Brouwer took his doctoral degree at the University of Amsterdam, the dissertation which he defended on that occasion is still of interest to us, both from a historical *and* from a philosophical-mathematical point of view. A recent republication of the dissertation,[1] as well as the publication of two biographies of Brouwer[2] bear witness to this general interest.

Despite the fact that Brouwer's intuitionistic mathematics does not play a leading part in the mainstream of the mathematical practice of today, the constructivistic founding, based on the ur-intuition alone, of his 'old' intuitionism from before, say, 1919 is still widely discussed and commented on. There certainly were constructive foundations of mathematics long before Brouwer,[3] but unique for Brouwer is the ur-intuition of successive and well-separated events, connected by a 'flowing', as the most fundamental basis possible for *every* mathematical construction.

Another unique feature of Brouwer's mature intuitionism from after 1917 (in which the ur-intuition remains the ultimate foundation) is, as the reader may know, formed by the 'two acts'.[4] We will see that the two acts are already present in the dissertation, albeit in a more or less latent form and not yet explicitly denoted as 'acts'.

Let us put first that there is not one main subject in Brouwer's dissertation, which has to be discussed; there is a number of topics in it, which ask for a reinterpretation, for a correction of a misinterpretation or even for a first interpretation; it concerns topics which are sometimes thoroughly discussed and sometimes merely touched upon by Brouwer. Just to mention a few: the exact construction of the integers and of the rationals, departing from the ur-intuition; another one is Brouwer's solution to the continuum problem or, a third one, the 'denumerably infinite unfinished' cardinality. A justified reinterpretation is often made possible by the recent discovery of a number of notebooks by Brouwer's hand, written during the time of preparation of his dissertation. The

[1] [Dalen 2001].

[2] One in Dutch, [Dalen, D. van 2001] and one scientific biography in English, of which the first volume has been published, [Dalen, D. van 1999].

[3] [Beth 1967], page 149, where for example Plotinus, Nicolaus Cusanus and Hobbes are mentioned.

[4] See page 57.

result of a reinterpretation more than once leads to the conclusion that the topic concerned is another example of *mathematics as a free creation of the human mind, solely based on the ur-intuition of the two-ity continuous and discrete*, which is the heart and the essence of Brouwer's summary on the last page of his dissertation. A second central item of this summarizing conclusion will be the very limited role that is granted to logic in the construction of mathematics; logic just plays its modest part in the language that describes this construction.

In our dissertation we will investigate the subjects concerning the most basic foundational matters from Brouwer's doctoral thesis, *On the foundations of mathematics*, which will lead to his intuitionistic mathematics of ten years later. This limited selection explains the omission of mathematical-technical matters like the thoroughly discussed group-theoretical foundation of the arithmetical operations on the measurable continuum, whereas the foundation and construction of the measurable continuum itself forms an important part of our dissertation. Also parts which we regard as today's common and commonly shared knowledge are very briefly discussed, just mentioned, or even completely omitted.

We realize that the last two phrases might give rise to some confusion about which dissertation we are sometimes referring to, and this brings us to the following introductory remark: since this work is a dissertation about a dissertation, a sharp distinction has to be made between *this* or *our dissertation* and *Brouwer's dissertation*. We will stick to this terminology unless the context makes it unambiguously clear which one we are talking about. In practice this will come down to the following: when, for instance, a reference is made to 'chapter 3' without further specification, or to 'our chapter 3', the third chapter of this dissertation is meant; in case we have the third chapter of Brouwer's dissertation is mind, this will be explicitly stated.

As said, the aim of this dissertation is to present a (re)interpretation of those topics in Brouwer's dissertation, which are of foundational interest for his future development. This concerns mainly the following subjects:

– The ur-intuition and the construction of the ω-scale, departing from that intuition,

– the status of 'signs',

– the construction of the η-scale,

– the scale of integers,

– the everywhere dense η-scale,

– the Bolzano-Weierstrass theorem,

– covering an everywhere dense scale with a continuum.

– the possible point sets, in particular the third construction rule for sets,

– the continuum problem and Brouwer's solution,

– Brouwer's view on (theoretical) physics and on natural sciences in general. We realize that this is outside the realm of pure foundational mathematics, but in view of the importance that Brouwer himself attached to this subject, we feel that we cannot get around it; it is too directly linked with his general view on mankind and on human society.

– objectivity and apriority,

– the role of logic in the construction of the mathematical building,

– the hypothetical judgement and its constructivistic interpretation,

– the possible cardinalities, in particular the 'denumerably infinite unfinished' cardinality,

– the denumerably unfinished cardinality of the set of mathematical theorems versus Gödel's first incompleteness theorem,

– the actual infinite and related problems like e.g. those of 'known/unknown' and 'finished/unfinished' of the (parts of) a mathematical construction.

For that purpose the arrangement of the chapters is as follows:

In the first chapter a general and concise survey of set theory will be presented, as known among mathematicians in the year 1907, the year of Brouwer's academic promotion. In this survey there will be an emphasis on the work of Cantor, with additional remarks on (for us relevant) parts of the work of Dedekind, Poincaré, Zermelo, Schoenflies and Bernstein.

In the subsequent chapters the content of Brouwer's dissertation will be discussed in regard to the following subjects:
Chapter 2: The ur-intuition of mathematics,
Chapter 3: The continuum,
Chapter 4: The possible point sets,
Chapter 5: The continuum problem and Brouwer's solution to it,
Chapter 6: Mathematics and Experience,
Chapter 7: The role of logic,
Chapter 8: The summary and the theses.
In an appendix Brouwer's own bibliography is presented, as pieced together from his dissertation and from the notebooks.

For the relevant research we have mainly used the republication (1981) of Brouwer's dissertation,[5] which edition also includes the Brouwer-Korteweg correspondence, the *Rejected Parts*, two reviews of the dissertation by Mannoury, the *Addenda and Corrigenda*[6] (1917) and the 1908-paper *The unreliability of the logical principles*.[7] As an additional source of information we were also able

[5] [Brouwer 1981].
[6] *Addenda en corrigenda over de grondslagen der wiskunde*, [Brouwer 1917a].
[7] *De onbetrouwbaarheid der logische principes*, [Brouwer 1908a].

to use, thanks to the mentioned recent new edition of Brouwer's dissertation,[8] the corrections by Brouwer's hand, made in his own copy of the dissertation in view of a reissue of it, in or shortly after 1917. For the English version of the dissertation, *On the foundations of mathematics*, we have mostly adopted the translation as this was published in the *Collected works*.[9] When quoting from the dissertation, we always refer to the original page numbers, which are added in the margin of both the Dutch and the English editions.

As a reference to Brouwer's published work we have adopted the codification from *A bibliography of L.E.J. Brouwer*.[10]

To gain a better insight into Brouwer's ideas on the several topics that will be discussed, but also into the process of development leading to his views, we have, apart from the dissertation itself, the following sources at our diposal:

1^{st}. *Brouwer's nine notebooks.*

About six months before the date of taking his doctoral degree Brouwer composed a synopsis of what seemed to be a rather random collection of notes about different mathematical and philosophical topics. This synopsis was known for a long time and, judging by its content and reference system, it had to refer to a set of notebooks, unknown until some five years ago.[11]

The notebooks, nine in number, which turned up later, matched perfectly the content and the references of the synopsis. A transcription of these notebooks is now completed, annotations are in preparation, and the whole will be published later.

The dating of the notebooks can be made within reasonable margins. On the first page of the first notebook a reference in the margin is made to the Heidelberger Congress, which took place in 1904. The proceedings of this congress were published in 1905, hence this year most likely marks the beginning of the notes.

On page 28 of the ninth and last notebook we find a reference to the *Revue de Métaphysique et de Morale* of the year 1906, which gives us a reliable indication of the end of the notes.

However, at the end of this last notebook, on page 32, Brouwer mentioned the second volume of *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten* by A. Schoenflies, which was published in 1908. And on the last page (page 33)

[8] [Dalen 2001].

[9] *L.E.J. Brouwer. Collected works*, [Brouwer 1975].

[10] [Dalen, D. van 1997].

[11] Most likely the synopsis is not a first draft for his dissertation. A little more than a month after the letter in which he announced the composition of the synopsis (7 September 1906), Brouwer wrote another letter (16 October 1906) to his thesis supervisor Korteweg, which contained the planned arrangement of the chapters for his dissertation on the basis of the notebooks and its synopsis. The first draft for this chapter arrangement shows some similarity with the final result in the dissertation and no similarity whatsoever with the chapter arrangement of the synopsis. Moreover, the synopsis does not contain fundamentally new insights, it just presents in a systematic way concise and summarizing remarks from pages of the notebooks, thereby referring to the relevant notebook pages.

there is a reference to two papers from Brouwer's own hand, *On the structure of perfect point sets*,[12], published in 1910, and *Zur Analysis Situs*, written in May 1909 and published in the *Mathematische Annalen*, also in 1910.[13]

Apparently Brouwer continued to make some notes for at least one year after the defence of his dissertation, which took place on 19 February 1907. Most likely the last blank pages of the last notebook were made useful, since on earlier pages not a single trace can be found of any note, definitely dating from later than the date of the defence.

As remarked already, Brouwer started composing the mentioned synopsis some six months before the public defence of his dissertation; this 'period of six months' can be concluded from the letter of 7 september 1906 from Brouwer to Korteweg:

> For some time I am now in Blaricum, where I can spend more efficiently all my time on my work. I stopped reading others and I am busy arranging my notes into chapters.
>
> I am feeling so much the stronger in my conviction, now that I perceive that I still hold as my view the notes of about two years ago, after all the reading since. Except that I now can support them better with the help of mathematical developments, than at that time.[14]

The justified conclusion seems to be that the notebooks were mainly written during the years 1905 and 1906, with some additional notes on the last two pages of the last notebook from the years 1907, 1908 and 1909.

The content of the notebooks consists of short notes, varying in length from one line to one page, discussing and commenting on a large variety of topics, like the foundations of mathematics, the foundations of (projective) geometry, philosophy, mysticism (often with pessimistic overtones),[15] but there are also long discussions on the continuum and on sets, which are the subjects of our main concern. Brouwer also made elaborate notes on potential theory; the results of those notes were published separately in the *Verslagen van de Koninklijke Nederlandse Akademie van Wetenschappen*.[16]

[12] Over de Structuur der Perfecte Puntverzamelingen; [Brouwer 1910b] and [Brouwer 1910a], published in the *Verslagen van de Koninklijke Nederlandse Akademie van Wetenschappen* and in the *Proceedings* of the same Academy, of 1910.

[13] [Brouwer 1910c].

[14] included in [Brouwer 1981] and [Dalen 2001]: Sinds enige tijd ben ik in Blaricum, waar ik beter al mijn tijd aan mijn werk kan geven. Met het lezen van anderen ben ik opgehouden, en ben nu bezig mijn aantekeningen te ordenen en onder hoofdstukken te brengen.

Ik voel mij des te sterker in mijn overtuiging, nu ik merk, mijn aantekeningen van ongeveer twee jaar geleden ook nu nog, na mijn lectuur van de tussentijd, geheel voor mijn rekening te kunnen nemen. Alleen kan ik ze nu beter met wiskundige ontwikkelingen steunen dan toen.

[15] To get an idea of Brouwer's mysticism and his pessimistic outlook on life and on humanity, see [Brouwer 1905]; it is also translated in English, see [Stigt 1996].

[16] The *Proceedings of the Royal Dutch Academy of Sciences*. In fact the Dutch version was published in the *KNAW Verslagen* and the English translation in the *KNAW Proceedings*.

To the modern reader it appears to be difficult to transform this totality of seemingly loose remarks, thought experiments and comments into a systematic overview of the developments in Brouwer's mathematical and philosophical thinking that we are about to investigate.

Especially in the last three notebooks Brouwer was very much concerned with the continuum, the admissible sets and their possible cardinalities. Whereas in the first three or four notebooks only a small number of paragraphs was devoted to sets and the continuum, from the second half of the sixth notebook onwards page-long discussions follow on these topics; apparently these pages must have been written during the year 1906.

Also we find in the notebooks many rudimentary ideas and thought experiments that return only in much later developments of his mathematical thinking.

We adopt Brouwer's convention to indicate the page references to the notebooks: III–7 refers to page 7 of the third notebook. This system of codification is Brouwer's own, as he used it in the synopsis of his notebooks. In this synopsis Brouwer composed from the seemingly random and chaotic abundance of comments and remarks a more systematic whole, by collecting the different subjects into chapters.

Finally we make the following comment on the notebooks: Brouwer frequently quoted from the writings of others (Cantor, Poincaré, Russell, Couturat and many others). It is striking that in many cases, in fact more often than not, the quotes are not exactly verbatim, despite the fact that Brouwer put them between quotation marks. He often composed the quotes himself by selecting parts from longer sentences (without using the modern convention of inserting '(...)' for deleted words or groups of words), or he added new words to make from part of a sentence a complete one. Brouwer either took that liberty or quoted by heart. Since in the dissertation the quotes are strictly verbatim, the ones in the notebooks are most likely just references and mnemonics for later use in the dissertation.

2^{nd}. *The correspondence with his thesis supervisor Korteweg.*

During the time of preparation of his dissertation and especially during the final stage of actually writing it all down, Brouwer was in frequent and close contact with his thesis supervisor Korteweg, via personal visits as well as via letters.[17] As a result of criticism from the side of Korteweg, several parts of Brouwer's draft text did not find their way into his dissertation. These 'rejected parts' have been preserved.

The earliest known correspondence dates back to a letter of 15 February 1906, concerning the extension of Brouwer's study grant. The more substantive part begins with a letter of 7 September 1906, through which a reasonable dating of the period of writing down the content of the synopsis could be made; the relevant quotation was given above.

[17] Included in [Brouwer 1981] and in [Dalen 2001].

3^{rd}. *The Rejected Parts of the manuscript.*

These consist of the parts of Brouwer's draft for his dissertation which, as remarked above, were rejected as a result of Korteweg's criticism.[18]

4^{th}. *Preserved notes concerning the public defence of the dissertation, i.e. Mannoury's and Barrau's opposition, and Brouwer's reply to it.*[19]

An academic 'promotion', i.e. the ceremony of taking one's doctoral degree, consists of a public defence of the dissertation against objections raised by the examining professors, but in the old days in Amsterdam the examination was opened with the *opposition from the floor*, that is, opponents from the audience were allowed to attack the dissertation and the theses. That the opponents concerned and the content of their opposition did not come as a complete surprise to the candidate, can be concluded from the Korteweg correspondence.[20]

Also the following lecture and paper by Brouwer, as well as a letter to De Vries, a review and an inaugural lecture, although they were given, held, sent, or read after obtaining his doctoral degree, are relevant to a proper understanding and interpretation of the subjects, especially when viewed as a transitional stage on his way to intuitionistic mathematics:

5^{th}. The Rome lecture *Die möglichen Mächtigkeiten*, held at the International Congress of Mathematicians in Rome,1908.[21]

6^{th}. *The unreliability of the logical principles*,[22] published in the *Tijdschrift voor Wijsbegeerte*.[23]

7^{th}. Brouwer's *letter to J. de Vries*, professor in mathematics at Utrecht University, 1907.[24]

8^{th}. *Mannoury's review of the dissertation and Brouwer's reply.*[25]

9^{th} *Intuitionism and Formalism*,

Brouwer's inaugural adress on the occasion of his professorship at the University of Amsterdam, 1912.[26]

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[18] Included in [Brouwer 1981] and in [Dalen 2001].

[19] Found as loose sheets in the last notebook. The Mannoury-part is included in [Brouwer 1981] and in [Dalen 2001].

[20] For a detailed description of the ceremony, see [Dalen, D. van 1999], page 118, 119.

[21] Atti IV Congr. Intern. Mat. Roma III, page 569 – 571. Also included in [Brouwer 1975], page 102 – 104.

[22] *De onbetrouwbaarheid der logische principes.* See [Brouwer 1908a]. Also included in [Brouwer 1981] and in [Dalen 2001]; English translation in [Brouwer 1975], page 107 – 111.

[23] Dutch *Journal of Philosophy.*

[24] Included in [Dalen 2001].

[25] See [Brouwer 1981] and [Dalen 2001]. In fact, Mannoury wrote two reviews; Brouwer replied to the one which was published in the *Nieuw Archief voor Wiskunde.*

[26] See [Brouwer 1981] and [Dalen 2001]; for English translation see [Brouwer 1975].

Occasionally some additional short quotes will be drawn from other writings, not mentioned in the given list and usually from a much later date.

For each subject the mentioned sources will be examined on their relevance, beginning with the dissertation and thereupon, if needed or desired as an elucidation or as an aid to a proper interpretation of the views expressed in the dissertation, the other sources. It will be evident that particularly the notebooks will offer us the opportunity to trace any possible development in fundamental notions, as there are the continuum, sets and their possible cardinalities, but also developments in future notions like choice sequences and the spread concept. For a proper understanding of these remarks and notes we often have to refer to much later work by Brouwer's hand.

A final remark about the translation of the quotes: for the English version of the quotations from the work of Brouwer we have mainly used, as point of departure, the translation as it was published in the mentioned *Collected Works* (C.W.), edited by A. Heyting. However, we have modified the translation when that was, in our opinion, considered necessary.

Translations by others (e.g. Mancosu, Dresden) of texts by Brouwer, but not included in the C.W., will be mentioned separately.[27]

Original texts in German or French are left untranslated.

[27] Many English translations of Brouwer's texts, as far as they were originally written in Dutch, can be found in [Heijenoort 1967], [Benacerraf and Putnam 1983], [Stigt 1990], [Mancosu 1998] or [Ewald 1999].