APPENDIX 3

Descartes, Elizabeth and Apollonius' Problem

The letters
On October 21 1643 Descartes' wrote to Pollot (Letter 47) that not long ago ('dernierement', p. 133, l. 18) he had suggested to the Princess Elizabeth to apply her algebraic skills to 'la question des trois cercles' (l. 18), and that he feared that the problem was too difficult for her. It appears that Pollot had informed him that Elizabeth was working on the problem and thought she had solved it 'supposing only one root' (cf. Letter 57, p. 154, l. 5). Descartes did not believe she could succeed on the basis of this assumption and he composed a letter (Letter 58) to the Princess with comments on the problem and his own solution. I will refer to it as 'Descartes' first letter'. On 17 November he sent it to Pollot in The Hague with a covering letter (Letter 57), asking Pollot to find out whether Elizabeth was willing to read his mathematical explanations or preferred to proceed with her own attempts. Pollot handed over the letter to Elizabeth, who read it and on 21 November sent to Descartes 'what she had done' ('ce que i'ay fait', p. 159, l. 15), with an elegant covering letter (Letter 59). Her solution of the problem is lost. Descartes reacted on it in a long letter (Letter 61) of 29 November, to which I will refer as 'Descartes' second letter'. There are no further references to the problem in the correspondence.

The problem
The 'problem of the three circles', more usually called the 'problem of Apollonius', concerns three given circles in the plane, and requires to find a circle that touches each of the given circles.[1] There may be up to eight such circles to three given ones. Descartes and Elizabeth tacitly assumed that the three given circles are in a triangular arrangement as in Figure 1, and that the required circle is located in the space between the three circles. With these assumptions there is only one solution to the problem.

The problem was mentioned and attributed to Apollonius by Pappus in Book VII, 11 of his *Collectio*;[2] the Apollonian treatise *The tangencies* in which it occurred (as final problem) is no longer extant. Pappus did not give Apollonius' solution, but his comments implied that it employed ruler and compass constructions only. The earliest known solution of the problem is by Viète who published

[1] For the history of the problem see Pappus, *Book 7 of the Collection*, transl. and comm. A. Jones, 2 vols. (New York: Springer, 1986), 'The tangencies', pp. 534–538, and H. Dörrie, *100 great problems of elementary mathematics, their history and solution*, transl. from German by D. Antin (New York: Dover, 1965), pp. 154–160. Accessible introductions to the mathematics involved can be found in R. Courant and H. Robbins, *What is mathematics? An elementary approach to ideas and methods* (Oxford: Oxford University Press, 1996), pp. 125–127, and H.S.M. Coxeter, 'The problem of Apollonius', *American Mathematical Monthly*, 75 (1968), 5–15.

[2] The first printed edition is *Pappi Alexandrini mathematicae collectiones a Federico Commandino Urbinate in latinum conversae at commentariis illustratae* (Pesaro 1588); cf. Pappus, *Book 7 of the Collection*, pp. 90–95.

it in a reconstruction of Apollonius' treatise.[3] Viète had earlier challenged Van Roomen to solve Apollonius' problem; but Van Roomen gave a solution by the intersection of conic sections,[4] whereas Viète had found a solution by ruler and compass. Most geometers at the time held that constructions in geometry should preferably be by ruler and compass. Viète's solution was entirely in the classical Greek style, it did not involve algebra. Thus at the time Descartes proposed Elizabeth to solve it, the 'problem of the three circles' was well known and recognized as difficult; no algebraic solution of it had been published.

Solutions

According to the predominant view of geometrical problem solving at the time, the solution of Apollonius' problem should consist of a geometrical construction by ruler and compass, or a sequence of such constructions, producing the centre and the radius of the required tangent circle. Neither Descartes nor Elizabeth produced such a construction. This was partly because of the complexity of calculations involved, but also because the two correspondents did not adopt the above view on solutions. In Descartes' opinion it was enough for solving a geometrical problem to decide whether it could be constructed by ruler and compass or needed higher-order means of construction. To do so one only needed to determine the degree of the relevant equation; if that degree was 1 or 2 the problem could be solved by ruler and compass, otherwise not. In his *Geometrie* he had explained in general how constructions could be derived from equations. Thus this part of problem solving involved no novelty for him; for individual problems he considered the further detailed elaboration of a construction from the equation as superfluous and boring, it did not 'serve to cultivate or recreate the mind but only to exercise the patience of some laborious calculator' (p. 158, ll. 91–93).

Elizabeth's aim in solving the problem was to derive, by means of letter algebra (i.e. calculations with letters standing for given but indeterminate quantities), a theorem about the configuration of one circle touching three given circles. We may tentatively relate this approach to her mathematical training by Stampioen who devoted a brief final section of his book on algebra of 1639 to what he called the 'revelation of the proofs' (*Openbaeringhe der Vertoogen*).[5] Here he showed how the proofs of certain theorems by Euclid and Viète could be derived by using letter algebra. The section is of interest because nowhere else in the book did Stampioen use letter algebra, always choosing explicit numbers for the given quantities in a problem. Descartes' comments on Elizabeth's work make clear that she used letter algebra as a matter of course, which shows, for the time, a considerable mastery of algebra.

[3] François Viète, *Apollonius Gallus seu exsuscitata Apollonii Pergaei* peri epafoon *geometria. Ad V.C. Adrianum Romanum Belgam* (Paris: Le Clerc, 1600).

[4] Adriaan van Roomen, *Problema Apolloniacum* (Würzburg 1596).

[5] Johan Stampioen de Jonge, *Algebra ofte nieuwe stel-regel waerdoor alles ghevonden wordt in de wis-kunst, wat vindtbaer is* (The Hague 1639).

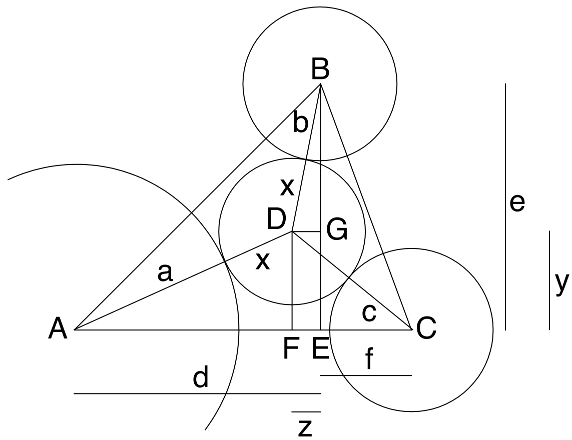

Figure 1: Descartes' approach; based on the figures in Letter 58, p. 156 (lower figure) and p. 158, emendated.

Descartes' approach

Descartes explained his approach to the problem in his first letter. I discuss it referring to figure 1, based on the Figures in Letter 58 (p. 156 (lower figure) and p. 158); the letters are the ones Descartes used in the text and the figure. A, B, C, D are the centres of the three given circles and the unknown tangent circle; the corresponding radii are a, b, c, x. DF and BE, drawn perpendicular to the base AC, are the heights of the triangles ABC and ADC respectively and DG is perpendicular to BE. The centres A, B and C are given, which means that the shape and size of triangle ABC is given too. Descartes decided to interpret this by choosing, beyond a, b and c, three more indeterminates, namely the height $e = BE$ and the two parts $d = AE$ and $f = EC$ in which E divides the basis AC.

Descartes used three unknowns in his algebraic calculations, namely the radius x of the required circle, and the segments $y = DF = GE$, and $z = DG = FE$. By applying Pythagoras' theorem to the right-angled triangles AFD, DGB and FCD he found (p. 157, ll. 63–65) the following three equations (in slightly modernized notation, but I keep to Descartes' habit of writing xx in stead of x^2 for squares).

$$aa + 2ax + xx = dd - 2dz + zz + yy \ , \quad (1)$$
$$bb + 2bx + xx = ee - 2ey + yy + zz \ , \quad (2)$$
$$cc + 2cx + xx = ff + 2fz + zz + yy \ . \quad (3)$$

He noted that subtracting Equation 1 from Equations 2 and 3 yields two equations without the terms xx, yy, zz, hence linear in x, y and z. He did not write these down explicitly. They are:

$$\begin{aligned} bb - aa + 2(b-a)x &= ee - dd - 2ey + 2dz \ , & (4) \\ cc - aa + 2(c-a)x &= ff - dd + 2(f+d)z \ . & (5) \end{aligned}$$

From these he derived expressions for z (p. 157, l. 79)) and y (p. 158, l. 87) in terms of x and the six indeterminates:

$$\begin{aligned} z &= \frac{1}{2}d - \frac{1}{2}f + \frac{cc - aa + 2cx - 2ax}{2d + 2f} \ , & (6) \\ y &= \frac{1}{2}e - \frac{bb}{2e} - \frac{bx}{e} - \frac{df}{2e} + \frac{ccd + aaf + 2cdx + 2afx}{2ed + 2ef} \ . & (7) \end{aligned}$$

Descartes then argued that if these expressions for z and y are substituted in any of the three equations 1, 2 and 3, a second-degree equation in the one remaining unknown x will be found. This can indeed be seen directly from the form of the expressions involved, and Descartes did not actually derive the equation. The fact that it would be of second degree in x indicated that the problem was 'plane' (p. 158, ll. 90–91), that is, it could (in principle) be constructed by ruler and compass.

It was clear enough how one should insert the values of z and y in one of the three equations, and in his *Geometrie* Descartes had given a general method to derive a ruler and compass construction from a second-degree equation.[6] Nevertheless his assertion that his proof of constructibility provided the essential answer to the problem obscures the fact that Descartes could hardly have derived the final equation, and even less the pertaining ruler and compass construction. Indeed the derivation of the final equation in x would be more than laborious—calculation with the computer algebra program *Mathematica* shows that this equation consists of no less than 87 terms, each of which is a product of six factors (for instance: b^4d^2, $4a^2cdfx$, $8de^2fx^2$).[7] It is practically impossible to apply the procedure explained in the *Geometrie* to such an equation, and even if it were, it would hardly be satisfactory as solution of a problem whose formulation is as straightforward as that of Apollonius' problem.

[6] AT VI, 374–376.

[7] This is the equation:

$$\begin{aligned} &b^4d^2 - 2b^2c^2d^2 + c^4d^2 + a^4e^2 - 2a^2c^2e^2 + c^4e^2 - 2a^2d^2e^2 - 2b^2d^2e^2 + \\ &d^4e^2 + d^2e^4 - 2a^2b^2df + 2b^4df + 2a^2c^2df - 2b^2c^2df + 2b^2d^3f - 2c^2d^3f - \\ &2a^2de^2f - 4b^2de^2f - 2c^2de^2f + 2d^3e^2f + 2de^4f + a^4f^2 - 2a^2b^2f^2 + b^4f^2 - \\ &2a^2d^2f^2 + 4b^2d^2f^2 - 2c^2d^2f^2 + d^4f^2 - 2b^2e^2f^2 - 2c^2e^2f^2 + 2d^2e^2f^2 + e^4f^2 - \\ &2a^2df^3 + 2b^2df^3 + 2d^3f^3 + 2de^2f^3 + d^2f^4 + e^2f^4 + 4b^3d^2x - 4b^2cd^2x - \\ &4bc^2d^2x + 4c^3d^2x + 4a^3e^2x - 4a^2ce^2x - 4ac^2e^2x + 4c^3e^2x - 4ad^2e^2x - 4bd^2e^2x - \\ &4a^2bdfx - 4ab^2dfx + 8b^3dfx + 4a^2cdfx - 4b^2cdfx + 4ac^2dfx - 4bc^2dfx + 4bd^3fx - \\ &4cd^3fx - 4ade^2fx - 8bde^2fx - 4cde^2fx + 4a^3f^2x - 4a^2bf^2x - 4ab^2f^2x + 4b^3f^2x - \\ &4ad^2f^2x + 8bd^2f^2x - 4cd^2f^2x - 4be^2f^2x - 4ce^2f^2x - 4adf^3x + 4bdf^3x + 4b^2d^2x^2 - \\ &8bcd^2x^2 + 4c^2d^2x^2 + 4a^2e^2x^2 - 8ace^2x^2 + 4c^2e^2x^2 - 4d^2e^2x^2 - 8abdfx^2 + 8b^2dfx^2 + \\ &8acdfx^2 - 8bcdfx^2 - 8de^2fx^2 + 4a^2f^2x^2 - 8abf^2x^2 + 4b^2f^2x^2 - 4e^2f^2x^2 = 0. \end{aligned}$$

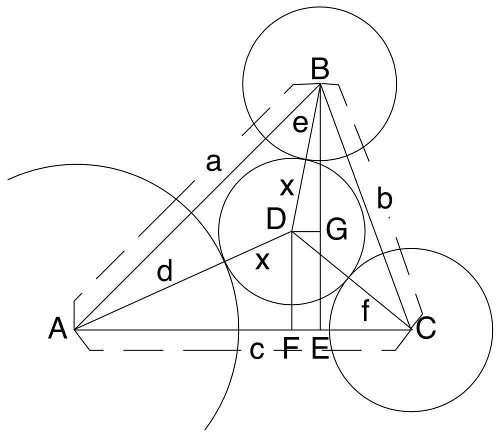

Figure 2: The previous figure with Elizabeth's probable lettering.

Several remarks in his letters show that Descartes was aware of the algebraic complexity of the problem. Why, then, did he propose the problem to the Princess? It was not particularly suitable for raising enthusiasm for his methods in an interested amateur, however intelligent. The most likely explanation seems to be that when he proposed the problem to Elizabeth, Descartes had not really applied himself to it and only later became aware of its complexity. Such a course of events is consistent with his worries about the Princess being caught in lengthy calculations (Letter 47, p. 133, ll. 17–21).

Elizabeth's approach

The summary of her calculations about the 'problem of the three circles' which Elizabeth sent to Descartes is lost, but her covering letter (Letter 59, 21 November 1643) and Descartes' comments on her work in his second letter allow us to reconstruct the main characteristics of her approach. As mentioned above, she applied the techniques of letter algebra with some confidence and her aim was to derive 'a theorem' about the circle tangent to three given circles. In this she failed, because, as she wrote, her results were insufficiently clear to detect such a theorem (p. 159, l. 10).

In his second letter Descartes changed his lettering, probably taking over Elizabeth's own (cf. p. 164, ll. 45–46, and Figures 1 and 2). Unlike Descartes, Elizabeth used only one unknown in her calculations, namely the radius of the required tangent circle; as indeterminates she chose the radii d, e, f of the three given circles, and the sides $a = AB$, $b = BC$, $c = AC$ of the triangle ABC. Then, evidently, she tried to derive an equation for the radius x of the tangent circle in terms of these indeterminates, in the hope that this equation could be interpreted geometrically as a theorem about the tangent circle of three given

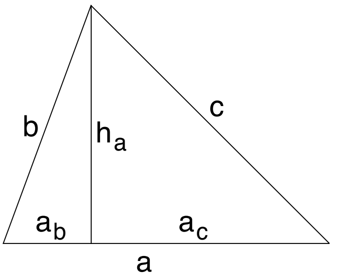

Figure 3: Triangle with altitude h_a and segments a_b and a_c along the base.

ones. The passage in Descartes' second letter in which he elaborates a simplified version of the problem claiming to follow the approach of the Princess (pp. 164–165, ll. 40–71), suggests that she reasoned along the following lines: The common shape of the two triangles ABC and ADC is represented in Figure 3 and it is not difficult to derive that:

$$a_b = \frac{a^2 + b^2 - c^2}{2a}, \tag{8}$$

$$a_c = \frac{a^2 - b^2 + c^2}{2a}, \tag{9}$$

$$h_a^2 = \frac{(a+b+c)(a+b-c)(a+c-b)(b+c-a)}{4a^2}. \tag{10}$$

These results imply that (see Figure 2) the segments AF, DF, FC, AE, BE, EC can directly be expressed in terms of a, b, c, d, e, f, and x, and the same applies, via subtraction, for the sides DG and BG of the right-angled triangle DGB, whose hypotenuse DG is equal to $e + x$. Applying the theorem of Pythagoras in this triangle will then produce an equation in x and the six indeterminates.

The relation 10 is actually equivalent to the theorem of Heron, which Descartes assumed Elizabeth knew (p. 156, l. 24), so this strategy could well have occurred to her. Could she have succeeded? The use of computer algebra shows that her strategy would lead to computational complications similar to those which Descartes would have encountered had he tried to go beyond the result that the problem was constructible by ruler and compass: the equation resulting from this line of reasoning is slightly simpler than the one Descartes' approach would yield (78 in stead of 87 terms) but the intermediate calculations are somewhat more extensive. So it is no wonder that, as she wrote, Elizabeth could not see clearly enough to detect a theorem.

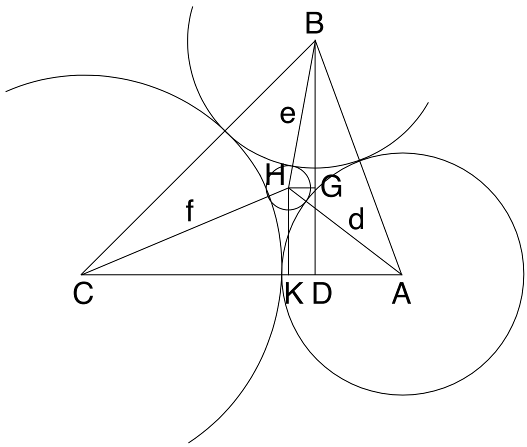

Figure 4: The special case in which the given circles touch each other.

The special case: the given circles touch each other
In his second letter Descartes then introduced a special, simpler (and geometrically appealing) case of the problem (pp. 164–165, ll. 50–71), and applied Elizabeth's own approach to it, apparently to show that the complications she had met were not essential but merely caused by the extreme algebraic complexity of the general problem. It was the case in which the three given circles are tangent to each other (see Figure 4). In that case the sides of the triangle can be expressed in the radii of the given circles:

$$AB = a = d + e \quad , \tag{11}$$
$$CB = b = e + f \quad , \tag{12}$$
$$CA = c = f + d \quad , \tag{13}$$

so the only three indeterminates left are d, e, and f. Descartes gave two results which the Princess should find in the intermediary calculations (l. 55), namely

$$AK = \frac{dd + df + dx - fx}{d + f} , \tag{14}$$
$$AD = \frac{dd + df + de - fe}{d + f} , \tag{15}$$

(they result from applying equation 8 to the triangles CHA and CBA of Figure 4). Then he gave the equation which should result (ll. 59–62):[8]

[8]The equation is related to the fact, independently discovered by several nineteenth-century geometers, that the curvatures ρ_i of four mutually tangent circles satisfy the equation

$$2(\rho_1^2 + \rho_2^2 + \rho_3^2 + \rho_4^2) = (\rho_1 + \rho_2 + \rho_3 + \rho_4)^2 \ . \tag{16}$$

$$ddeeff + ddeexx + ddffxx + eeffxx =$$
$$= 2deffxx + 2deefxx + 2ddefxx + 2deeffx + 2ddeffx + 2ddeefx \,. \quad (17)$$

Because Elizabeth had been searching for a 'theorem', Descartes expressed the meaning of the equation in words (ll. 63–66). Remarkably he did not attempt to give a geometrical meaning to these words; note particularly that, although the terms of the equation are products, he calls them 'sums,' and thereby avoids the problem of dimensions involved in geometrically interpreting products of six factors which are line segments. Finally he applied the theorem in a numerical example in which $d = 2$, $e = 3$ and $f = 4$ (ll. 67–71).

It is not clear how exactly Descartes arrived at equation 17.[9]

Discussions of mathematical strategies

The two letters of Descartes contain several passages on the best way to approach geometrical problems. These passages are of interest because the Princess' efforts in solving the 'problem of the three circles' made Descartes change, or at least adjust his opinions. The main theme was how best to translate a geometrical problem into algebra, in particular how to choose the unknowns and the indeterminates such that the algebraic calculations would be easiest.

It appears that in their earliest discussions about geometrical problem solving Descartes had stressed the advantages of introducing more than one unknown (cf. p. 155, ll. 3–5 and p. 154, l. 5). Yet Elizabeth had used only one unknown. In his first letter, written after receiving this information through Pollot, he explained his reasons for preferring more than one unknown and also for choosing unknowns and indeterminates along two perpendicular directions (pp. 155–156, ll. 6–22). In that way, he claimed, one needed only the simplest geometrical theorems (namely similarity of triangles and Pythagoras' theorem) in translating a geometrical problem into algebraic terms, after which all but

(The curvature of a circle is defined as the inverse of its radius. Substituting $x = 1/\rho_1$, $d = 1/\rho_2$, $e = 1/\rho_3$, $f = 1/\rho_4$ in the equation and reordering yields the result.) In 1960 A. Aeppli called attention to the relation of Equation 16 to the equation in Descartes' second letter to Elizabeth. Since then the result on the curvatures of four mutually tangent circles is known as 'the Descartes Circle Theorem', although there are no indications that Descartes realized this implication of his result. Cf. D. Pedoe, 'On a theorem in geometry', *American Mathematical Monthly*, 74 (1967), pp. 627–640, esp. p. 634.

[9]The most direct way I have found is to express Descartes' indeterminates d, e, f in terms of the radii a, b, c of the given circles:

$$d = \frac{a^2 + ac + ab - bc}{a + c}, \quad e = \frac{2\sqrt{abc(a + b + c)}}{a + c}, \quad f = \frac{c^2 + bc + ac - ab}{a + c}$$

(using the fact that the given circles touch each other), inserting these expressions in Descartes' results 1– 3, 6 and 7, and determining the final equation by inserting the resulting values for z and y in one of the equations 1– 3. The calculations involved are considerable, but still within the range of Descartes' power and patience. However, in his second letter Descartes gave the impression that he followed Elizabeth's rather than his own approach and in that case the derivation of the equation is somewhat less straightforward; although still not unsurmountable.

one of these unknowns could be eliminated by straightforward algebraic techniques. Working with one unknown only, and with mutually oblique indeterminates, would involve the use of more complicated geometrical theorems in order to express the elements of the problem into algebra. Descartes was aware that the elimination techniques in the first approach were the algebraic equivalents of the more complex geometrical theorems in the second; the advantage was that the eliminating techniques were more straightforward.

Descartes illustrated these ideas in his first letter by an approach which he then probably thought Elizabeth might have taken (p. 156, ll. 23–36). This approach (see Figure 2) consisted in introducing as single unknown the radius x of the tangent circle and deriving algebraic expressions for the areas of the four triangles ABC, ADC, ADB, BDC by Heron's theorem ('un Theoreme qui enseigne à trouver l'aire d'un triangle par ses trois costez', l. 24; Descartes does not mention Heron), which asserts that a triangle with sides a, b, c has area $\sqrt{s(s-a)(s-b)(s-c)}$, with $s = \frac{1}{2}(a+b+c)$. The sides of the four triangles are easily expressed in terms of the given radii d, e f, the given sides a, b, c of triangle ABC, and the unknown radius x of the required tangent circle. Equating the sum of the areas of the three smaller triangles to the area of triangle ABC then yields an equation involving x as only unknown. Descartes warns that this approach would lead to many 'superfluous multiplications' (l. 35). Indeed, for deriving the final equation one needs to remove four square roots, which enormously complicates the resulting equation. This difficulty did not occur in Descartes' own approach (as he could foresee without actually bringing it to an end) and the complications involved in an approach Elizabeth might possibly have chosen could well make Descartes worry about her confrontation with labyrinthine calculations.

Descartes' preference for choosing the line segments to be used in the equations along two perpendicular directions is evident in his own approach, in which (see above) he chose AE, EB and EC as indeterminates, rather than, as in the approach based on Heron's theorem, the oblique sides AB, BC, CA of the triangle ABC. (One notes that his choice is equivalent to taking the origin of a rectangular coordinate system in E; the segments AE, EC, EB, FE, FD are in effect the coordinates of the centres of the three given and the one required circle.)

Descartes formulated the opinions mentioned above in his first letter, when the only thing he knew about Elizabeth's approach was that she had used one rather than several unknowns. When he received her letter and her report on her investigation of the problem of the three circles he was much impressed and even adjusted his view on choosing indeterminates. Indeed the second letter is different in tone from the first; Elizabeth is now addressed as a fellow mathematician rather than a prospective pupil. The letter is full of praise and positive surprise. Descartes is pleased to see that Elizabeth's 'calcul' was entirely similar to his own (p. 163, ll. 5–6). He praises her patience in calculating (ll. 12–15), and her technique of representing complicated expressions by single letters (ll. 17–18).

The letter also provides enough detail about Elizabeth's results to conclude that Descartes' positive reaction was more than due politeness in addressing a person of rank. Contrary to his expectations, he did not find Elizabeth much hampered by her choice of only one unknown. Moreover, she had chosen the sides of triangle ABC as indeterminates, a choice which Descartes had rejected because they were not along perpendicular directions. Now he realized an important advantage of that choice, namely that the resulting formulas were symmetric in a, b, c and in d, e, f respectively ('Car les trois lettres a, b, c, y sont disposées en mesme façon, et aussi les trois d, e, f' (ll. 48–49)). This made him change his mind and he acknowledged the supremacy of her approach in this respect ('car il est meilleur, pour cela, que celui que i'avois proposé' (ll. 46–47)). Indeed the formulas in his own approach (cf. Equations 6 and 7) markedly lack symmetry because of the asymmetrical choice of his indeterminates AE, BE and CE. In his subsequent solution of the simplified problem he actually points out the symmetry of Equation 17 ('les quatre lettres d, e, f, x, qui estant les rayons des quatre cercles, ont semblables rapport l'une à l'autre' (ll. 52–53)), and Equations 14 and 15 ('x est dans la ligne AK, comme e dans la ligne AD').

Descartes accepts Elizabeth's aim of deriving a theorem as a valid alternative to his own aim of determining the constructibility of a problem. Such a theorem, he writes, may serve as general rule for solving many problems of the same kind ('qui serve de regle generale pour en soudre plusieurs autres semblables' (l. 28)). He also explains that in deriving such a theorem one has to resubstitute expressions which for ease of calculating one has represented by single letters, whereas if one, like Descartes, only wishes to determine constructibility, such a resubstitution is not necessary. At the end of the letter he actually presents these two aims as equivalent approaches, the one best served by Elizabeth's approach with one unknown and a symmetrical choice of indeterminates, the other by his own approach with more than one unknown and indeterminates along two perpendicular directions. This final clear formulation of the two different approaches (ll. 72–78) suggests that the discussion with the Princess has contributed positively to Descartes' own understanding of the relation between the aims and the techniques of solving geometrical problems by algebra.

[HB]